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ON THE NUMBER OF SOLUTIONS IN POSITIVE INTEGERS OF THE EQUATION $yz + zx + xy = n$.

BY L. J. MORDELL.

About sixty years ago, Liouville,* in commenting upon a paper by , stated some results concerning the number of solutions in positive ...s of the equation

$$yz + zx + xy = n, \tag{1}$$

where, of course, n is a positive integer. For example, if x, y, z are odd and $y + z \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$, the number of solutions is represented by $F(n)$, the number of uneven binary classes of determinant $-n$. Again if $n \equiv 1 \pmod 4$ and y and z are both odd, the number of solutions is $F(n)$ plus one half of the number of divisors of n. The following ...ation of the meaning of the symbol $F(n)$ may make these results ...telligible to those who have not studied the theory of numbers. ... quadratic forms

$$a\xi^2 + 2b\xi\eta + c\eta^2$$

...lly denoted by (a, b, c) where a, b, c are any integers satisfying

$$b^2 - ac = -n, \qquad a > 0,$$

... be grouped in a finite number of classes such that the forms in a class ... be transformed into each other by a linear substitution

$$\xi' = p\xi + q\eta, \qquad \eta' = r\xi + s\eta$$

...ere p, q, r, s are integers satisfying the equation

$$ps - qr = 1.$$

Moreover, the forms in two different classes can not be transformed into ... other by such a substitution. The total number of classes is called ...n). Representatives of these classes are selected in a particular way† ...d referred to as reduced forms. We call the classes, in which a and c are ...t simultaneously even, the uneven classes, and denote the number thereof ... $F(n)$. In reckoning these class numbers, we adopt the usual convention, ...t a class $(k, 0, k)$ is reckoned as $\frac{1}{2}$, and a class $(2k, k, 2k)$ as $\frac{1}{3}$.

* Jour. de maths., series 2, tome 7, 1862, page 44.

† See for example Mathews's "Theory of Numbers," pp. 59–73.

Only a few months ago, Prof. E. T. Bell in his paper * "Class Numbers and the Form $yz + zx + xy$" proved that the number of solutions of equation (1) is equal to $3G(n) - 3$ if n is a prime. He stated that his method, which depends upon formulæ of the type introduced by Liouville into Analysis, also gives the results for n composite, but that he has not published them as they are rather complicated.

I shall now show that the number of solutions of equation (1), when no restrictions are made upon x, y, z, n, except that they are all positive integers, is equal to $3G(n)$, provided that a solution in which one of the unknowns is zero is reckoned as $\frac{1}{2}$ instead of 1. For example, if $n = 19$, the number of solutions is 12; six solutions arising from the permutations of 1, 3, 4, three from 1, 1, 9, and three from the six permutations of 0, 1, 19, since each solution is now reckoned as $\frac{1}{2}$. Also there are four classes of binary forms of determinant -19 represented by (1, 0, 19), (2, 1, 10), $(4, \pm 1, 5)$, so that the formula is verified. It of course includes Bell's result as a particular case, since he has not adopted the convention for the solutions with one of the unknowns equal to zero.

Consider separately the solutions for which $x + y$ is odd or even. In the former case put

$$\begin{aligned} 2x &= 2m + 1 + t, \\ 2y &= 2m + 1 - t, \end{aligned}$$

so that t is an odd number and

$$0 < |t| \leq 2m + 1,$$

and m is a positive integer or zero. The equation (1) becomes

$$\begin{aligned} 4n &= (2m + 1)^2 - t^2 + 4z(2m + 1) \\ &= (2m + 1)(2m + 1 + 4z) - t^2, \end{aligned} \tag{2}$$

where $m, z = 0, 1, 2, \cdots$,

$$0 < |t| \leq 2m + 1$$

and the convention applies to the solutions for which either $|t| = 2m + 1$ or $z = 0$. The number of solutions of (2) is equal to $F(4n)$, as we can establish a unique correspondence between them and the uneven classes of binary quadratics of determinant $-4n$. For corresponding to any solution, we have the quadratic form of determinant $-4n$,

$$(2m + 1, \quad t, \quad 2m + 1 + 4z). \tag{3}$$

But the uneven classes of determinant $-4n$ can be represented by the

* *Tôhoku Mathematical Journal*, Vol. 19, May, 1921, pp. 105–116.

forms (a, b, c) with

$$b^2 - ac = -4n,$$

where $c \geq a \geq 2|b|$ and c and a are not both even; if any of the equality signs hold, we take only the positive values of b. These forms can be arranged in three groups according to the residues of $a, b, c \pmod 2$.

In group I, a, b, c are all odd, whence $a \equiv c \pmod 4$. In group II, a is odd, b is even, c is even; and in group III, a is even, b is even, c is odd. Now the number of solutions of (2) in which $|t| \leq m$ is obviously equal to the number of forms in the first group, since when $z = 0$, the solution is reckoned as $\frac{1}{2}$, that is the forms $(2m + 1, t, 2m + 1)$, are reckoned only when t is positive. For the solutions with $|t| > m$ we consider instead of the quadratic (3), the form derived from it, by changing x into $x \mp y$ according as $t \gtrless 0$, namely,

$$[2m + 1, t \mp (2m + 1), (4m + 2 + 4z \mp 2t)] = (A, B, C) \text{ say.} \qquad (4)$$

Hence $A > 2|B|$ and either the form (A, B, C) or $(C, -B, A)$ is reduced, so that the number of solutions now is equal to the number of forms in the groups (2) and (3). We note that when $|t| = 2m + 1$, the convention concerning zero solutions means that the form $(2m + 1, 0, 4z)$ is only reckoned once. This proves that when $x + y$ is odd, the number of solutions of (1) is $F(4n)$ and this is also equal to $2F(n)$.

When $x + y$ is even, we put $x = m + t$, $y = m - t$, so that (1) becomes

$$n = m^2 - t^2 + 2mz, \qquad (5)$$

where

$$0 \leq |t| \leq m, \quad \text{and} \quad m, z = 0, 1, 2, 3, \cdots$$

with the convention when either $z = 0$ or $|t| = m$.

The solutions of this equation can be found in exactly the same way as in (2); but I have already done this in my paper "On Class Relation Formulæ"* and the conventions there adopted concerning the number of solutions are exactly the same as the present ones. Two cases are considered. When m is odd, the number of solutions† is $F(n)$; when m is even, the number of solutions‡ is $3G(n) - 3F(n)$. Hence the total number of solutions of (5) is $3G(n) - 2F(n)$, and adding these to $2F(n)$, the number of solutions of (4), we have the final result, that the number of solutions in positive integers of equation (1) is $3G(n)$ provided we reckon only half the solutions when one of the unknowns is zero.§

* *Messenger of Mathematics*, Vol. 46, 1916.

† Page 133 of the above.

‡ Page 134 of the above.

§ A very simple proof has since been given by Whitehead in the *Proceedings of the London Mathematical Society*, records of proceedings at meetings, etc., issued May 30, 1922.

We have also shown * that

(1) $x + y$ is odd for $2F(n)$ of these solutions,
(2) $x + y \equiv 2 \pmod 4$ for $F(n)$ of these solutions,
(3) $x + y \equiv 0 \pmod 4$ for $3G(n) - 3F(n)$ of these solutions.†

We may note also that when n is not a perfect square

$$x + y \equiv 1 \pmod 4 \text{ for } F(n) \text{ of these solutions,}$$
$$x + y \equiv 3 \pmod 4 \text{ for } F(n) \text{ of these solutions.}$$

For if m is even in equation (2) the binary quadratics (3) represent odd numbers of the form $4k + 1$, and hence are half of the total number of odd forms of determinant $-4n$, as follows from the elementary properties of the generic character of binary quadratic forms.

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* These include Liouville's results.

† Also given by Lerch in the *Rozpravy ceske Akad. Prague*, 7, 1898, No. 4 [Bohemian].

A CLOSED SET OF NORMAL ORTHOGONAL FUNCTIONS.*

By J. L. Walsh.

Introduction.

A set of normal orthogonal functions $\{\chi\}$ for the interval $0 \leqq x \leqq 1$ has been constructed by Haar,† each function taking merely one constant value in each of a finite number of sub-intervals into which the entire interval (0, 1) is divided. Haar's set is, however, merely one of an infinity of sets which can be constructed of functions of this same character. It is the object of the present paper to study a certain new closed set of functions $\{\varphi\}$ normal and orthogonal on the interval (0, 1); each function φ has this same property of being constant over each of a finite number of sub-intervals into which the interval (0, 1) is divided. In fact each function φ takes only the values $+1$ and -1, except at a finite number of points of discontinuity, where it takes the value zero.

The chief interest of the set φ lies in its similarity to the usual (e.g., sine, cosine, Sturm-Liouville, Legendre) sets of orthogonal functions, while the chief interest of the set χ lies in its *dissimilarity* to these ordinary sets. The set φ shares with the familiar sets the following properties, none of which is possessed by the set χ: the nth function has $n-1$ zeroes (or better, sign-changes) interior to the interval considered, each function is either odd or even with respect to the mid-point of the interval, no function vanishes identically on any sub-interval of the original interval, and the entire set is uniformly bounded.

Each function χ can be expressed as a linear combination of a finite number of functions φ, so the paper illustrates the changes in properties which may arise from a simple orthogonal transformation of a set of functions.

In § 1 we define the set χ and give some of its principal properties. In § 2 we define the set φ and compare it with the set χ. In § 3 and § 4 we develop some of the properties of the set φ, and prove in particular that every continuous function of bounded variation can be expanded in terms of the φ's and that every continuous function can be so developed in the sense not of convergence of the series but of summability by the first Cesàro mean. In § 5 it is proved that there exists a continuous function which

* Presented to the American Mathematical Society, Feb. 25, 1922.

† *Mathematische Annalen*, Vol. 69 (1910), pp. 331–371; especially pp. 361–371.

cannot be expanded in a convergent series of the functions φ. In § 6 there is studied the nature of the approach of the approximating functions to the sum function at a point of discontinuity, and in § 7 there is considered the uniqueness of the development of a function.

§ 1. Haar's Set χ.

Consider the following set of functions:

$$f_0(x) \equiv 1, \quad 0 \leqq x \leqq 1,$$

$$f_1^{(1)}(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x \leqq 1, \end{cases} \qquad f_1^{(2)}(x) \equiv \begin{cases} 1, & \frac{1}{2} < x \leqq 1, \\ 0, & 0 \leqq x < \frac{1}{2}, \end{cases}$$

. . . .

$$f_k^{(i)}(x) \equiv \begin{cases} 1, \dfrac{i-1}{2^k} < x < \dfrac{i}{2^k}, & i = 1, 2, 3, \cdots, 2^k, \\ 0, \ 0 \leqq x < \dfrac{i-1}{2^k}, \text{ or } \dfrac{i}{2^k} < x \leqq 1, & k = 1, 2, 3, \cdots, \infty; \end{cases}$$

these functions may be defined at a point of discontinuity to have the average of the limits approached on the two sides of the discontinuity.

If we have at our disposal all the functions $f_k^{(i)}$, it is clear that we can approximate to any continuous function in the interval $0 \leqq x \leqq 1$ as closely as desired and hence that we can expand any continuous function in a uniformly convergent series of functions $f_k^{(i)}$. For a continuous function $F(x)$ is uniformly continuous in the interval $(0, 1)$, and thus uniformly in that entire interval can be approximated as closely as desired by a linear combination of the functions $f_k^{(i)}$ where k is chosen sufficiently large but fixed. The approximation can be made better and better and thus will lead to a uniformly convergent series of functions $f_k^{(i)}$.

Haar's set χ may be found by normalizing and orthogonalizing the set $f_k^{(i)}$, those functions to be ordered with increasing k, and for each k with increasing i. The set χ consists of the following functions:*

$$\chi_0(x) \equiv 1, \quad 0 \leqq x \leqq 1, \qquad \chi_1(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leqq 1, \end{cases}$$

$$\begin{array}{llll} \chi_2^{(1)}(x) = \sqrt{2}, & \chi_2^{(2)} = 0, & 0 \leqq x < \frac{1}{4}, \\ \quad = -\sqrt{2}, & \quad = 0, & \frac{1}{4} < x < \frac{1}{2}, \\ \quad = 0, & \quad = \sqrt{2}, & \frac{1}{2} < x < \frac{3}{4}, \\ \quad = 0, & \quad = -\sqrt{2}, & \frac{3}{4} < x \leqq 1, \end{array}$$

. . . .

* L. c., p. 361.

$$
\begin{aligned}
\chi_n^{(k)} &= \sqrt{2^{n-1}}, && \frac{k-1}{2^{n-1}} < x < \frac{2k-1}{2^n}, && k = 1, 2, 3, \cdots, 2^{n-1},\\
&= -\sqrt{2^{n-1}}, && \frac{2k-1}{2^n} < x < \frac{k}{2^{n-1}}, && n = 1, 2, 3, \cdots, \infty,\\
&= 0, && 0 < x < \frac{k-1}{2^{n-1}} \quad \text{or} \quad \frac{k}{2^{n-1}} < x < 1.
\end{aligned}
$$

The same convention as to the value of $\chi_n^{(k)}$ at a point of discontinuity is made as for the $f_n^{(k)}$, and $\chi_n^{(k)}(0)$ and $\chi_n^{(k)}(1)$ are defined as the limits of $\chi_n^{(k)}$ as x approaches 0 and 1.

For any particular value of N, all the functions $f_n^{(k)}$, $n < N$, can be expressed linearly in terms of the functions $\chi_n^{(k)}$, $n < N$, and conversely.

Let $F(x)$ be any function integrable and with an integrable square in the interval (0, 1); its formal development in terms of the functions χ is

$$
\begin{aligned}
F(x) \sim \chi_0(x) \int_0^1 F(y)\chi_0(y)dy + \chi_1(x) \int^1 F(y)\chi_1(y)dy + \cdots \\
+ \chi_n^{(k)}(x) \int_0^1 F(y)\chi_n^{(k)}(y)dy + \cdots.
\end{aligned}
\tag{1}
$$

This series (1) is formed with coefficients determined formally as for the Fourier expansions, and it is well known that $S_m(x)$, the sum of the first m terms of this series, is that linear combination $F_m(x)$ of the first m of the functions χ which renders a minimum the integral

$$\int_0^1 (F(x) - F_m(x))^2 dx.$$

That is, $S_m(x)$ is in the sense of least squares the best approximation to $F(x)$ which can be formed from a linear combination of the first m functions χ; it is likewise true that $S_m(x)$ is the best approximation to $F(x)$ which can be formed from a linear combination of those functions $f_n^{(k)}$ that are dependent on the first m functions χ.

Let $F(x)$ be continuous in the closed interval (0, 1). If ϵ is any positive number, there exists a corresponding number n such that

$$|F(x') - F(x'')| < \epsilon \qquad \text{whenever} \qquad |x' - x''| < \frac{1}{2^n}.$$

We interpret $S_{2^n}(x)$ as a linear combination of the functions $f_n^{(k)}$. The multiplier of the function $f_n^{(k)}$ which appears in $S_{2^n}(x)$ is chosen so as to furnish the best approximation in the interval $\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ to the function $F(x)$, so it is evident that $S_{2^n}(x)$ approximates to $F(x)$ uniformly in the entire interval (0, 1) with an approximation better than ϵ. The function

$S_{2^n+i}(x)$ cannot differ from $F(x)$ by more than ϵ at any point of the interval (0, 1), and so for all the functions $S_{2^n+i}(x)$. Thus we have

THEOREM I. *If $F(x)$ is continuous in the interval (0, 1), series (1) converges uniformly to the value $F(x)$ if the terms are grouped so that each group contains all the 2^{n-1} terms of a set $\chi_n^{(k)}$, $k = 1, 2, 3, \cdots, 2^{n-1}$.*

Haar proves that the series actually converges uniformly to $F(x)$ without the grouping of terms,* and establishes many other results for expansions in terms of the set χ; to some of these results we shall return later.

§ 2. The Set φ.

The set φ, which it is the main purpose of this paper to study, consists of the following functions:

$$\varphi_0(x) \equiv 1, \qquad 0 \leqq x \leqq 1, \qquad \varphi_1(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \leqq 1, \end{cases}$$

$$\varphi_2^{(1)}(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{4},\ \frac{3}{4} < x \leqq 1, \\ -1, & \frac{1}{4} < x < \frac{3}{4}, \end{cases}$$

$$\varphi_2^{(2)}(x) \equiv \begin{cases} 1, & 0 \leqq x < \frac{1}{4},\ \frac{1}{2} < x < \frac{3}{4}, \\ -1, & \frac{1}{4} < x < \frac{1}{2},\ \frac{3}{4} < x \leqq 1, \end{cases}$$

. . . .

$$\begin{aligned} \varphi_{n+1}^{(2k-1)}(x) &\equiv \begin{cases} \varphi_n^{(k)}(2x), & 0 \leqq x < \frac{1}{2}, \\ (-1)^{k+1}\varphi_n^{(k)}(2x-1), & \frac{1}{2} < x \leqq 1, \end{cases} \\ \varphi_{n+1}^{(2k)}(x) &\equiv \begin{cases} \varphi_n^{(k)}(2x), & 0 \leqq x < \frac{1}{2}, \\ (-1)^{k}\varphi_n^{(k)}(2x-1), & \frac{1}{2} < x \leqq 1, \end{cases} \\ k = 1, 2, 3, \cdots, 2^{n-1}, &\qquad n = 1, 2, 3, \cdots, \infty. \end{aligned} \tag{2}$$

In general, the function $\varphi_n^{(1)}$, $n > 0$, is to be used, with the horizontal scale reduced one half and the vertical scale unchanged, to form the functions $\varphi_{n+1}^{(1)}$ and $\varphi_{n+1}^{(2)}$ in each of the halves $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$ of the original interval; the function $\varphi_{n+1}^{(1)}(x)$ is to be even and the function $\varphi_{n+1}^{(2)}$ odd with respect to the point $x = \frac{1}{2}$. Similarly the function $\varphi_n^{(k)}$ is to be used to form the functions $\varphi_{n+1}^{(2k-1)}$ and $\varphi_{n+1}^{(2k)}$, the former of which is even and the latter odd with respect to the point $x = \frac{1}{2}$. All the functions $\varphi_n^{(k)}$ are to be taken positive in the interval $\left(0, \dfrac{1}{2^n}\right)$. The function $\varphi_n^{(k)}$ is to be defined at points of discontinuity as were the functions f and χ, and at $x = 0$ to have the value 1, and at $x = 1$ to have the value $(-1)^{k+1}$.† The function

* L. c., p. 368.

† If it is desired to develop periodic functions by means of the set φ [or the similar sets f and χ] simultaneously in all the intervals $\cdots$, $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $\cdots$, it will be wise to change these definitions at $x = 0$ and $x = 1$ so that always the value of $\varphi_n^{(k)}(x)$ is the arithmetic mean of the limits approached at these points to the right and to the left.

$\varphi_n^{(k)}$ is odd or even with respect to the point $x = \frac{1}{2}$ according as k is even or odd.

The functions φ_0, φ_1, $\varphi_2^{(1)}$, $\varphi_2^{(2)}$ have 0, 1, 2, 3 zeroes (i.e., sign-changes) respectively interior to the interval (0, 1). The function $\varphi_{n+1}^{(2k-1)}(x)$ has twice as many zeroes as the function $\varphi_n^{(k)}$; and $\varphi_{n+1}^{(2k)}(x)$ has one more zero, namely at $x = \frac{1}{2}$, than has $\varphi_{n+1}^{(2k-1)}(x)$. Thus the function $\varphi_n^{(k)}$ has $2^{n-1} + k - 1$ zeroes; this formula holds for $n = 2$ and follows for the general case by induction. Hence each function $\varphi_n^{(k)}$ has one more zero than the preceding; the zeroes of these functions increase in number precisely as do the zeroes of the classical sets of functions—sine, cosine, Sturm-Liouville, Legendre, etc. We shall at times find it convenient to use the notation $\varphi_0, \varphi_1, \varphi_2, \cdots$ for the functions $\varphi_n^{(k)}$; the subscript denotes the number of zeroes.

The orthogonality of the system φ is easily established. Any two functions $\varphi_n^{(k)}$ are orthogonal if $n < 3$, as may be found by actually testing the various pairs of functions. Let us assume this fact to hold for $n = 1, 2, 3, \cdots, N-1$; we shall prove that it holds for $n = N$. By the method of construction of the functions φ, each of the integrals

$$\int_0^{1/2} \varphi_N^{(k)}(x)\varphi_m^{(l)}(x)dx, \qquad \int_{1/2}^1 \varphi_N^{(k)}(x)\varphi_m^{(l)}(x)dx, \qquad\qquad m \leqq N,$$

is the same except possibly for sign as an integral

$$\int_0^1 \varphi_{N-1}^{(j)}(y)\varphi_{m-1}^{(l)}(y)dy$$

after the change of variable $y = 2x$ or $y = 2x - 1$. Each of these two integrals [in fact, they are the same integral] whose variable is y has the value zero, so we have the orthogonality of $\varphi_N^{(k)}(x)$ and $\varphi_m^{(l)}(x)$:

$$\int_0^1 \varphi_N^{(k)}(x)\varphi_m^{(l)}(x)dx = 0.$$

This proof breaks down if the two functions $\varphi_{N-1}^{(j)}(y)$, $\varphi_{m-1}^{(l)}(y)$ are the same, but in that case either $\varphi_N^{(k)}(x)$ and $\varphi_m^{(l)}(x)$ are the same and we do not wish to prove their orthogonality, or one of the functions $\varphi_N^{(k)}(x)$, $\varphi_m^{(l)}(x)$ is odd and the other even, so the two are orthogonal.

Each of the functions $\varphi_n^{(k)}(x)$ is normal, for we have

$$|\varphi_n^{(k)}(x)| \equiv 1$$

except at a finite number of points.

Each of the functions $\chi_0, \chi_1, \chi_2^{(1)}, \chi_2^{(2)}, \cdots, \chi_{n+1}^{(2^n)}$ can be expressed linearly in terms of the functions $\varphi_0, \varphi_1, \varphi_2^{(1)}, \varphi_2^{(2)}, \cdots, \varphi_{n+1}^{(2^n)}$. Thus for $n = 1$ we have

$$\chi_0 = \varphi_0, \quad \chi_1 = \varphi_1, \quad \chi_2^{(1)} = \tfrac{1}{2}\sqrt{2}(\varphi_2^{(1)} + \varphi_2^{(2)}), \quad \chi_2^{(2)} = \tfrac{1}{2}\sqrt{2}(-\varphi_2^{(1)} + \varphi_2^{(2)}).$$

It is true generally that except for a constant normalizing factor $\sqrt{2}$, the function $\chi_{n+1}^{(k)}$, $k \leqq 2^{n-1}$, is the same linear combination of the functions $\frac{1}{2}[\varphi_{n+1}^{(2k-1)} + \varphi_{n+1}^{(2k)}]$ as is $\chi_n^{(k)}$ of the functions $\varphi_n^{(k)}$, and the function $\chi_{n+1}^{(k)}$, $k > 2^{n-1}$, is the same linear combination of the functions $\frac{1}{2}(-1)^{k+1}[\varphi_{n+1}^{(2k-1)} - \varphi_{n+1}^{(k)}]$ as is $\chi_n^{(k-2^{n-1})}$ of the functions $\varphi_n^{(k)}$.

It is similarly true that all the functions $\varphi_0, \varphi_1, \cdots, \varphi_{n+1}^{(2^n)}$ can be expressed linearly in terms of the functions $\chi_0, \chi_1, \cdots, \chi_{n+1}^{(2^n)}$. Thus we have for $n = 2$,

$$\varphi_0 = \chi_0, \qquad \varphi_1 = \chi_1, \qquad \varphi_2^{(1)} = \tfrac{1}{2}\sqrt{2}(\chi_2^{(1)} - \chi_2^{(2)}), \qquad \varphi_2^{(2)} = \tfrac{1}{2}\sqrt{2}(\chi_2^{(1)} + \chi_2^{(2)}).$$

The general fact appears by induction from the very definition of the functions φ.

The set χ is known to be closed;* it follows from the expression of the χ in terms of the φ that the set φ is also closed.

The definition of the functions $\varphi_n^{(k)}$ enables us to give a formula for $\varphi_n^{(k)}(x)$. Let us set, in binary notation,

$$x = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots, \qquad a_i = 0 \text{ or } 1.$$

If x is a binary irrational or if in the binary expansion of x there exists $a_i \neq 0$, $i > n$, the following formulas hold for $\varphi_n^{(k)}$:

$$\begin{array}{ll}
\varphi_0 = 1, & \varphi_1 = (-1)^{a_1}, \\
\varphi_2^{(1)} = (-1)^{a_1+a_2}, & \varphi_2^{(2)} = (-1)^{a_2}, \\
\varphi_3^{(1)} = (-1)^{a_2+a_3}, & \varphi_3^{(2)} = (-1)^{a_1+a_2+a_3}, \\
\varphi_3^{(3)} = (-1)^{a_1+a_3}, & \varphi_3^{(4)} = (-1)^{a_3}, \\
\varphi_4^{(1)} = (-1)^{a_3+a_4}, & \varphi_4^{(2)} = (-1)^{a_1+a_3+a_4}, \\
\varphi_4^{(3)} = (-1)^{a_1+a_2+a_3+a_4}, & \varphi_4^{(4)} = (-1)^{a_2+a_3+a_4}, \\
\varphi_4^{(5)} = (-1)^{a_2+a_4}, & \varphi_4^{(6)} = (-1)^{a_1+a_2+a_4}, \\
\varphi_4^{(7)} = (-1)^{a_1+a_4}, & \varphi_4^{(8)} = (-1)^{a_4}, \\
\cdots\cdots & \cdots\cdots, \\
\cdots\cdots & \cdots\cdots.
\end{array} \tag{3}$$

The general law appears from these relations; always we have

$$\begin{aligned} \varphi_n^{(1)} &= (-1)^{a_{n-1}+a_n}, \\ \varphi_n^{(k)} &= \varphi_{k-1}\varphi_n^{(1)}. \end{aligned} \tag{4}$$

A general expression for $\varphi_n^{(k)}(x)$ when x is a binary rational can readily be computed from formulas (3), for we have expressions for the values of $\varphi_n^{(k)}$ for neighboring larger and smaller values of the argument than x.

* That is, there exists no non-null Lebesgue-integrable function on the interval (0, 1) which is orthogonal to all functions of the set; l. c., p. 362.

§ 3. Expansions in Terms of the Set $\{\varphi\}$.

The following theorem results from Theorem I by virtue of the remark that all the functions $\varphi_n^{(k)}$ can be expressed in terms of the functions $\chi_n^{(i)}$ and conversely, and from the least squares interpretation of a partial sum of a series of orthogonal functions:

THEOREM II. *If $F(x)$ is continuous in the interval* (0, 1), *the series*

$$F(x) \sim \varphi_0(x) \int_0^1 F(y)\varphi_0(y)dy + \varphi_1(x) \int_0^1 F(y)\varphi_1(y)dy + \cdots \varphi_i^{(j)}(x) \int_0^1 F(y)\varphi_i^{(j)}(y)dy + \cdots, \tag{5}$$

converges uniformly to the value $F(x)$ if the terms are grouped so that each group contains all the 2^{n-1} terms of a set $\varphi_n^{(k)}$, $k = 1, 2, 3, \cdots, 2^{n-1}$.

Series (5) after the grouping of terms is precisely the same as series (1) after the grouping of terms.

Theorem II can be extended to include even discontinuous functions $F(x)$; we suppose $F(x)$ to be integrable in the sense of Lebesgue. Let us introduce the notation

$$F(a+0) = \lim_{\epsilon=0} F(a+\epsilon), \qquad F(a-0) = \lim_{\epsilon=0} F(a-\epsilon), \qquad \epsilon > 0,$$

and suppose that these limits exist for a particular point $x = a$. We introduce the functions

$$F_1(x) = \begin{cases} F(x), & x < a, \\ F(a-0), & x \geqq a, \end{cases} \qquad F_2(x) = \begin{cases} F(a+0), & x \leqq a, \\ F(x), & x > a, \end{cases} \tag{6}$$

The least squares interpretation of the partial sums $S_{2^n}(x)$ of the series (1) or (5) as expressed in terms of the $f_i^{(j)}$ gives the result that if $h_1 < F(x) < h_2$ in any interval, then also $h_1 < S_{2^n}(x) < h_2$ in any completely interior interval if n is sufficiently large. It follows that $F_1(x)$ is closely approximated at $x = a$ by its partial sum S_{2^n} if n is sufficiently large, and that this approximation is uniform in any interval about the point $x = a$ in which $F_1(x)$ is continuous. A similar result holds for $F_2(x)$.

The function $F_1(x) + F_2(x)$ differs from the original function $F(x)$ merely by the function

$$G(x) = \begin{cases} F(a+0), & x < a, \\ F(a-0), & x > a. \end{cases}$$

The representation of such functions by sequences of the kind we are considering will be studied in more detail later (§ 6), but it is fairly obvious that such a function is represented uniformly except in the neighborhood

of the point a. If $F(x)$ is continuous at and in the neighborhood of a, or if a is dyadically rational, the approximation to $G(x)$ is uniform at the point a as well. Thus we have

THEOREM III. *If $F(x)$ is any integrable function and if $\lim_{x=a} F(x)$ exists for a point a, then when the terms of the series* (5) *are grouped as described in Theorem II, the series so obtained converges for $x = a$ to the value $\lim_{x=a} F(x)$. If $F(x)$ is continuous at and in the neighborhood of a, then this convergence is uniform in a neighborhood of a.*

If $F(x)$ is any integrable function and if the limits $F(a-0)$ and $F(a+0)$ exist for a dyadically rational point $x = a$, then the series with the terms grouped converges for $x = a$ to the value $\frac{1}{2}[F(a+0) + F(a-0)]$; this convergence is uniform in the neighborhood of the point $x = a$ if $F(x)$ is continuous on two intervals extending from a, one in each direction.

It is now time to study the convergence of series (5) when the terms are not grouped as in Theorems II and III. We shall establish

THEOREM IV. *Let the function $F(x)$ be of limited variation in the interval $0 \leqq x \leqq 1$. Then the series* (5) *converges to the value $F(x)$ at every point at which $F(a+0) = F(a-0)$ and at every point at which $x = a$ is dyadically rational. This convergence is uniform in the neighborhood of $x = a$ in each of these cases if $F(x)$ is continuous in two intervals extending from a, one in each direction.*

Since $F(x)$ is of limited variation, $F(a+0)$ and $F(a-0)$ exist at every point a. Theorem IV tacitly assumes $F(x)$ to be defined at every point of discontinuity a so that $F(a) = \frac{1}{2}[F(a+0) + F(a-0)]$.

Any such function $F(x)$ can be considered as the difference of two monotonically increasing functions, so the theorem will be proved if it is proved merely for a monotonically increasing function. We shall assume that $F(x)$ is such a function, and positive. We are to evaluate the limit of

$$\int_0^1 F(y) K_n^{(k)}(x, y) dy,$$

$$K_n^{(k)}(x, y) = \varphi_0(x)\varphi_0(y) + \varphi_1(x)\varphi_1(y) + \cdots + \varphi_n^{(k)}(x)\varphi_n^{(k)}(y).$$

We have already evaluated this limit for the sequence $k = 2^{n-1}$, so it remains merely to prove that

$$\lim_{n=\infty} \int_0^1 F(y) Q_n^{(k)}(x, y) dy = 0, \tag{7}$$

$$Q_n^{(k)}(x, y) = \varphi_n^{(1)}(x)\varphi_n^{(1)}(y) + \varphi_n^{(2)}(x)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(k)}(x)\varphi_n^{(k)}(y),$$

whatever may be the value of k.

We shall consider the function $F(x)$ merely at a point $x = a$ of con-

tinuity; that is, we study essentially the new functions F_1 and F_2 defined by equations (6). In the sequel we suppose a to be dyadically irrational; the necessary modifications for a rational can be made by the reader.

The following formulas are easily found by the definition of the $Q_n^{(k)}$; both x and y are supposed dyadically irrational:

$$Q_2^{(1)}(x, y) = \pm 1,$$

$$Q_2^{(2)}(x, y) = \begin{cases} 0 \text{ if } x < \tfrac{1}{2},\, y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\, y < \tfrac{1}{2}, \\ \pm 2 \text{ if } x < \tfrac{1}{2},\, y < \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\, y > \tfrac{1}{2}, \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q_n^{(1)}(x, y) = \pm 1,$$

$$Q_n^{(2)}(x, y) = \begin{cases} 0 \text{ if } x < \tfrac{1}{2},\, y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\, y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(1)}(2x, 2y) \text{ if } x < \tfrac{1}{2},\, y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(1)}(2x-1, 2y-1) \text{ if } x > \tfrac{1}{2},\, y > \tfrac{1}{2}, \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q_n^{(2k)}(x, y) = \begin{cases} 0 \text{ if } x < \tfrac{1}{2},\, y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\, y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(k)}(2x, 2y) \text{ if } x < \tfrac{1}{2},\, y < \tfrac{1}{2}, \\ 2Q_{n-1}^{(k)}(2x-1, 2y-1) \text{ if } x > \tfrac{1}{2},\, y > \tfrac{1}{2}, \end{cases}$$

$$Q_n^{(2k+1)}(x, y) = \begin{cases} \pm 1 \text{ if } x < \tfrac{1}{2},\, y > \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\, y < \tfrac{1}{2}, \\ \dfrac{Q_n^{(2k)} + Q_n^{(2k+2)}}{2} \text{ if } x < \tfrac{1}{2},\, y < \tfrac{1}{2} \text{ or if } x > \tfrac{1}{2},\, y > \tfrac{1}{2}. \end{cases}$$

The integral in (7) for $x = a$ is to be divided into three parts. Consider an interval bounded by two points of the form $x = \dfrac{\rho}{2^\nu}$, $x = \dfrac{\rho+1}{2^\nu}$, where ρ and ν are integers and such that

$$\frac{\rho}{2^\nu} < a < \frac{\rho+1}{2^\nu}.$$

Then we have

$$\begin{aligned} \int_0^1 F_1(y) Q_n^{(k)}(a, y)\,dy = \int_0^{\rho/2^\nu} F_1(y) Q_n^{(k)}(a, y)\,dy \\ + \int_{\rho/2^\nu}^{(\rho+1)/2^\nu} F_1(y) Q_n^{(k)}(a, y)\,dy + \int_{(\rho+1)/2^\nu}^1 F_1(y) Q_n^{(k)}(a, y)\,dy. \end{aligned} \tag{8}$$

These integrals on the right need separate consideration.

Let us set

$$\frac{\rho}{2^\nu} = \frac{\mu_1}{2^1} + \frac{\mu_2}{2^2} + \frac{\mu_3}{2^3} + \cdots + \frac{\mu_\nu}{2^\nu}, \qquad \mu_i = 0 \text{ or } 1.$$

The first integral in the right-hand member of (8) can be written

$$\int_0^{\mu_1/2^1} + \int_{\mu_1/2^1}^{(\mu_1/2^1)+(\mu_2/2^2)} + \cdots \int_{(\rho/2^\nu)-(\mu_\nu/2^\nu)}^{\rho/2^\nu} F_1(y) Q_n^{(k)}(a, y)\,dy. \tag{9}$$

Each of these integrals is readily treated. Thus, on the interval $0 \leqq y \leqq \frac{\mu_1}{2^1}$, $Q_n^{(k)}(a, y)$ takes only the values ± 1 or 0, is 0 if k is even and has the value $\pm \varphi_n^{(k)}(y)$ if k is odd. It is of course true that

$$\lim_{n=\infty} \int_0^1 \Phi(y)\varphi_n^{(k)}(y)dy = 0 \tag{10}$$

no matter what may be the function $\Phi(y)$ integrable in the sense of Lebesgue and with an integrable square.* Hence we have

$$\lim_{n=\infty} \int_0^{\mu_1/2^1} F_1(y)Q_n^{(k)}(a, y)dy = 0.$$

On the interval $\frac{\mu_1}{2^1} \leqq y \leqq \frac{\mu_1}{2^1} + \frac{\mu_2}{2^2}$, the function $Q_n^{(k)}(a, y)$ takes only the values $0, \pm 1, \pm 2$, and except for one of these numbers as constant factor, has the value $\varphi_n^{(k)}(y)$. It is thus true that

$$\lim_{n=\infty} \int_{\mu_1/2^1}^{(\mu_1/2^1)+(\mu_2/2^2)} F_1(y)Q_n^{(k)}(a, y)dy = 0.$$

From the corresponding result for each of the integrals in (9) and a similar treatment of the last integral in the right-hand member of (8), we have

$$\left.\begin{aligned} \lim_{n=\infty} \int_0^{\rho/2^\nu} F_1(y)Q_n^{(k)}(a, y)dy = 0, \\ \lim_{n=\infty} \int_{(\rho+1)/2^\nu}^1 F_1(y)Q_n^{(k)}(a, y)dy = 0. \end{aligned}\right\} \tag{11}$$

We shall obtain an upper limit for the second integral in (8) by the second law of the mean. We notice that

$$\left| \int_\xi^{(\rho+1)/2^\nu} Q_n^{(k)}(a, y)dy \right| \leqq \tfrac{1}{2},$$

whatever may be the value of ξ. In fact, this relation is immediate if n

* This well-known fact follows from the convergence of the series

$$\Sigma\, (a_n^{(k)})^2,$$

proved from the inequality

$$\int_0^1 (\Phi(x) - a_0\varphi_0 - a_1\varphi_1 - a_2^{(1)}\varphi_2^{(1)} - \cdots - a_n^{(k)}\varphi_n^{(k)})^2 dx \geqq 0,$$

where $a_n^{(k)} = \int_0^1 \Phi(y)\varphi_n^{(k)}(y)dy$.

is small and it follows for the larger values of n by virtue of the method of construction of the $Q_n^{(k)}$. Moreover, if $n \geqq \nu$ and if $\xi = \frac{\rho}{2^\nu}$, this integral has the value zero. We therefore have from the second law of the mean, $n \geqq \nu$,

$$\int_{\rho/2^\nu}^{(\rho+1)/2^\nu} F_1(y) Q_n^{(k)}(a, y)dy = F_1\left(\frac{\rho}{2^\nu}\right)\int_{\rho/2^\nu}^{\xi} Q_n^{(k)}(a, y)dy$$

$$+ F_1\left(\frac{\rho+1}{2^\nu}\right)\int_{\xi}^{(\rho+1)/2^\nu} Q_n^{(k)}(a, y)dy$$

$$= \left[F_1(a) - F_1\left(\frac{\rho}{2^\nu}\right)\right]\int_{\xi}^{(\rho+1)/2^\nu} Q_n^{(k)}(a, y)dy.$$

By a proper choice of the point $\frac{\rho}{2^\nu}$ we can make the factor of this last integral as small as desired; the entire expression will be as small as desired for sufficiently large n. The relations (11) are independent of the choice of $\frac{\rho}{2^\nu}$, so (7) is completely proved for the function F_1. A similar proof applies to F_2, so (7) can be considered as completely proved for the original function $F(x)$.

The uniform convergence of (5) as stated in Theorem IV follows from the uniform continuity of $F(x)$ and will be readily established by the reader.

§ 4. Further Expansion Properties of the Set φ.

The least square interpretation already given for the partial sums and the expression of the φ's in terms of the f's show that if the terms of (5) are grouped as in Theorems II and III, the question of convergence or divergence of the series at a point depends merely on that point and the nature of the function $F(x)$ in the neighborhood of that point. This same fact for series (5) when the terms are not grouped follows from (8) and (10) if $F(x)$ is integrable and with an integrable square. We shall further extend this result and prove:

THEOREM V. *If $F(x)$ is any integrable function, then the convergence or divergence of the series (5) at a point depends merely on that point and on the behavior of the function in the neighborhood of that point. If in particular $F(x)$ is of limited variation in the neighborhood of a point $x = a$, and if a is dyadically rational or if $F(a - 0) = F(a + 0)$, then series (5) converges for $x = a$ to the value $\frac{1}{2}[F(a - 0) + F(a + 0)]$. If $F(x)$ is not only of limited variation but is also continuous in two neighborhoods one on each side of a, and if a is dyadically rational or if $F(a - 0) = F(a + 0)$, the convergence of (5) is uniform in the neighborhood of the point a.*

Theorem V follows immediately from the reasoning already given and from (10) proved without restriction on Φ; we state the theorem for any bounded normal orthogonal set of functions ψ_n:

THEOREM VI. *If $\{\psi_n(x)\}$ is a uniformly bounded set of normal-orthogonal functions on the interval* (0, 1), *and if $\Phi(x)$ is any integrable function, then*

$$\lim_{n=\infty} \int_0^1 \Phi(x)\psi_n(x)dx = 0. \tag{12}$$

Denote by E the point set which contains all points of the interval for which $|\Phi(x)| > N$; we choose N so large that

$$\int_{\mathrm{E}} |\Phi(x)| dx < \epsilon,$$

where ϵ is arbitrary. Denote by E_1 the point set complementary to E; then we have

$$\int_0^1 \Phi(x)\psi_n(x)dx = \int_{\mathrm{E}} \Phi(x)\psi_n(x)dx + \int_{\mathrm{E}_1} \Phi(x)\psi_n(x)dx.$$

It follows from the proof of (10) already indicated that the second integral on the right approaches zero as n becomes infinite. The first integral is in absolute value less than $M\epsilon$ whatever may be the value of n, where M is the uniform bound of the ψ_n. It therefore follows that these two integrals can be made as small as desired, first by choosing ϵ sufficiently small and then by choosing n sufficiently large.*

It is interesting to note that Theorem VI breaks down if we omit the hypothesis that the set ψ_n is uniformly bounded. In fact Theorem VI does not hold for Haar's set χ. Thus consider the function

$$\Phi(x) = (x - \tfrac{1}{2})^{-\nu}, \qquad \nu < 1.$$

We have

$$\int_0^1 \Phi(x)\chi_n^{(2^{n-2}+1)}(x)dx = \sqrt{2^{n-1}} \int_{1/2}^{1/2+1/2^n} (x - \tfrac{1}{2})^{-\nu}dx$$

$$- \sqrt{2^{n-1}} \int_{1/2+1/2^n}^{1/2+1/2^{n-1}} (x - \tfrac{1}{2})^{-\nu}dx = \frac{(2^{n-1})^{\nu-(1/2)}}{1-\nu}[2^\nu - 1].$$

Whenever $\nu \geqq \frac{1}{2}$, it is clear that (12) cannot hold, and if $\nu > \frac{1}{2}$, there is a sub-sequence of the sequence in (12) which actually becomes infinite.

* Theorem VI is proved by essentially this method for the set $\psi_n(x) = \sqrt{2} \sin n\pi x$ by Lebesgue, *Annales scientifiques de l'école normale supérieure*, ser. 3, Vol. XX, 1903. See also Hobson, *Functions of a Real Variable* (1907), p. 675, and Lebesque, *Annales de la Faculté des Science de Toulouse*, ser 3, Vol. I (1909), pp. 25–117, especially p. 52.

We turn now from the study of the convergence of such a series expansion as (5) to the study of the summability of such expansions, and are to prove

THEOREM VII. *If $F(x)$ is continuous in the closed interval $(0, 1)$, the series (5) is summable uniformly in the entire interval to the sum $F(x)$.*

If $F(x)$ is integrable in the interval $(0, 1)$, and if $F(a-0)$ and $F(a+0)$ exist, and if either $F(a-0) = F(a+0)$ or a is dyadically rational, then the series (5) is summable for $x = a$ to the value $\frac{1}{2}[F(a-0) + F(a+0)]$. If $F(x)$ is continuous in the neighborhood of the point $x = a$, or if a is dyadically rational and $F(x)$ continuous in the neighborhood of a except for a finite jump at a, the summability is uniform throughout a neighborhood of that point.

In this theorem and below, the term *summability* indicates summability by the first Cesàro mean.

We shall find it convenient to have for reference the following

LEMMA. *Suppose that the series*

$$(b_1 + b_2 + \cdots + b_{n_1}) + (b_{n_1+1} + b_{n_1+2} + \cdots + b_{n_2}) + \cdots \\ + (b_{n_k+1} + b_{n_k+2} + \cdots + b_{n_{k+1}}) + \cdots \tag{13}$$

converges to the sum B and that the sequence

$$b_1, \quad \frac{2b_1 + b_2}{2}, \quad \frac{3b_1 + 2b_2 + b_3}{3}, \quad \cdots \quad \frac{(n_1 - 1)b_1 + (n_1 - 2)b_2 + \cdots + b_{n_1-1}}{n_1 - 1},$$
$$\frac{(n_1 - 1)b_1 + \cdots + b_{n_1-1}}{n}, \quad \frac{(n_1 - 1)b_1 + (n_1 - 2)b_2 + \cdots + b_{n_1-1} + b_{n_1+1}}{n_1 + 1}, \tag{14}$$
$$\frac{(n_1 - 1)b_1 + \cdots + b_{n_1-1} + 2b_{n_1+1} + b_{n_1+2}}{n_1 + 2}, \quad \cdots$$
$$\frac{(n_1 - 1)b_1 + \cdots + b_{n_1-1} + (n_2 - n_1 - 1)b_{n_1+1} + (n_2 - n_1 - 2)b_{n_1+2} + \cdots + b_{n_2-1}}{n_2 - 1},$$
$$\cdots,$$

converges to zero. Then the series

$$b_1 + b_2 + b_3 + \cdots \tag{15}$$

is summable to the sum B.

This lemma involves merely a transformation of the formulas involving the limit notions. Insert zeroes in series (13) so that the parentheses are respectively the n_1-th, n_2-th, n_3-th terms of the new series; this new series

converges to the sum B and hence is summable to the sum B. The term-by-term difference of the new series and (15) is the series

$$b_1 + b_2 + \cdots + b_{n_1-1} - (b_1 + b_2 + \cdots + b_{n_1-1}) + b_{n_1+1} + b_{n_1+2} + \cdots + b_{n_2-1} - (b_{n_1+1} + b_{n_1+2} + \cdots + b_{n_2-1}) + \cdots, \tag{16}$$

which is to be shown to be summable to the sum zero. The sequence corresponding to the summation of (16) is precisely (14).

A sufficient condition for the convergence to zero of (14) is that we have independently of m,

$$\lim_{k=\infty} \frac{mb_{n_k+1} + (m-1)b_{n_k+2} + \cdots + b_{n_k+m}}{m} = 0; \qquad m \leqq n_{k+1} - n_k, \tag{17}$$

for from a geometric point of view each term of the sequence (14) is the center of gravity of a number of terms such as occur in (17), each term weighted according to the number of b_i that appear in it. An (ϵ, δ)-proof can be supplied with no difficulty.

For the case of Theorem VII let us assume $F(x)$ integrable and that $F(a-0)$ and $F(a+0)$ exist. The series (15) is to be identified with the series (5), and (13) with (5) after the terms are grouped as in Theorem III. The sum that appears in (17) is, then, for $x = a$,

$$\frac{1}{m}\int_0^1 [m\varphi_n^{(1)}(a)\varphi_n^{(1)}(y) + (m-1)\varphi_n^{(2)}(a)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(m)}(a)\varphi_n^{(m)}(y)]F(y)dy, \quad m \leqq 2^{n-1}. \tag{18}$$

We shall prove that (18) formed for the function $F_1(y)$ defined in (6) and for a dyadically irrational has the limit zero as n becomes infinite.

Let us notice that

$$\frac{1}{m}\int_0^1 |m\varphi_n^{(1)}(a)\varphi_n^{(1)}(y) + (m-1)\varphi_n^{(2)}(a)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(m)}(a)\varphi_n^{(m)}(y)|dy = 1. \tag{19}$$

This follows directly from (3) and (4). The value of the integral in (19) is unchanged if we replace a by any dyadic irrational b. Choose $0 < b < 2^{-n}$, so that all the functions $\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{m-1}$ are positive for $x = b$. Then the integrand in (19) can be reduced merely to $m\varphi_0(y)$, so (19) is proved.

Let us consider the integral (18) formed for the function $F_1(y)$ to be divided as in (8), where as before

$$\frac{\rho}{2^\nu} < a < \frac{\rho+1}{2^\nu},$$

and let us denote by (20), (21), (22), (23) respectively the entire integral and its three parts. Then (22) can be made as small as desired simply by proper choice of the point $\frac{\rho}{2^\nu}$, for in the interval $\left(\frac{\rho}{2^\nu}, \frac{\rho+1}{2^\nu}\right)$ we can make $|F_1(y) - F_1(a)|$ uniformly small, we have established (19), and we have also

$$\int_{\rho/2^\nu}^{(\rho+1)/2^\nu} [m\varphi_n^{(1)}(a)\varphi_n^{(1)}(y) + (m-1)\varphi_n^{(2)}(a)\varphi_n^{(2)}(y) + \cdots + \varphi_n^{(m)}(a)\varphi_n^{(m)}(y)]F_1(a)dy = 0$$

if merely $n > \nu$.

The integral (21) is the average of m integrals of the type that appear in (8):

$$\int_0^{\rho/2^\nu} F_1(y)Q_n^{(k)}(a, y)dy, \qquad k = 1, 2, \cdots, m.$$

Thus the entire integral (21) approaches zero as n becomes infinite. Treatment in a similar way of the integral (23) proves that (20) approaches zero. It is likewise true that (18) formed for the function $F_2(y)$ also approaches zero as n becomes infinite. This completes the proof of the second sentence in Theorem VII for a dyadic irrational; we omit the proof for a dyadic rational. The uniformity of the continuity of $F(x)$ gives us readily the remaining parts of Theorem VII.

§ 5. Not Every Continuous Function Can Be Expanded in Terms of the φ.

The summability of the expansions of continuous functions in terms of the functions φ is another point of resemblance of those functions to the Fourier sine and cosine functions. Still another point of resemblance which we shall now establish is that there exists a continuous function whose expansion in terms of the φ's does not converge at every point of the interval.

Our proof rests on a beautiful theorem due to Haar,* by virtue of which the existence of such a continuous function will be shown if we prove merely that

$$\int_0^1 |K_n^{(k)}(a, y)|dy \tag{24}$$

is not bounded uniformly for all n and k. The point a is a point of divergence of the expansion of the continuous function and for our particular case may be chosen any point of the interval (0, 1). We shall study (24) in detail merely for a dyadically irrational; the integral (24) is independent of the point a chosen if a is dyadically irrational.

* L. c., p. 335. This condition holds for any set of normal orthogonal functions and is necessary as well as sufficient, if a slight restriction is added.

The integral (24) is bounded uniformly for all the values n if $k = 2^{n-1}$, so it will be sufficient to consider the integral

$$c_n^{(k)} = \int_0^1 |Q_n^{(k)}(a, y)|\, dy.$$

The following table shows the value of $c_n^{(k)}$ for small values of n and for each value of k:

$$\begin{array}{ll}
n = 2 & 1 \quad 1 \\
n = 3 & 1 \quad 1 \quad 1\tfrac{1}{2} \quad 1 \\
n = 4 & 1 \quad 1 \quad 1\tfrac{1}{2} \quad 1 \quad 1\tfrac{3}{4} \quad 1\tfrac{1}{2} \quad 1\tfrac{3}{4} \quad 1 \\
n = 5 & 1,\ 1,\ 1\tfrac{1}{2},\ 1,\ 1\tfrac{3}{4},\ 1\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1,\ 1\tfrac{7}{8},\ 1\tfrac{3}{4},\ 2\tfrac{1}{8},\ 1\tfrac{1}{2},\ 2\tfrac{1}{8},\ 1\tfrac{3}{4},\ 1\tfrac{7}{8},\ 1, \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

We have the general formulas

$$\begin{aligned}
c_n^{(1)} &= c_n^{(2n+1)} = 1, \\
c_n^{(k)} &= c_{n+1}^{(2k)}, \\
c_{n+1}^{(2k+1)} &= \tfrac{1}{2}\left[c_n^{(k)} + c_n^{(k+1)}\right] + \tfrac{1}{2},
\end{aligned}$$

so the $c_n^{(k)}$ are not uniformly bounded.

Theorem VIII. *If a point a is arbitrarily chosen, there will exist a continuous function whose φ-development does not converge at a.*

§ 6. The Approximation to a Function at a Discontinuity.

We have considered in § 3 and § 4 with a fair degree of completeness the nature of the approach to $F(x)$ of the formal development of an arbitrary function $F(x)$ in the neighborhood of a point of continuity of $F(x)$. We shall now consider the approach to $F(x)$ of this formal development in the neighborhood of a point of discontinuity of $F(x)$. We study this problem merely for a function which is constant except for a single discontinuity, a finite jump, but this leads directly to similar results for any function $F(x)$ at an isolated discontinuity which is a finite jump, if $F(x)$ is of such a nature that the expansion of $F(x)$ would converge uniformly in the neighborhood of the point of discontinuity were that discontinuity removed by the addition of a function constant except for a finite jump.

Let us consider the function

$$f(x) = \begin{cases} 1, & 0 \leqq x < a, \\ 0, & a < x \leqq 1. \end{cases}$$

If a is dyadically rational, $f(x)$ can be expressed as a finite sum of functions φ,* and thus is represented uniformly, if we make the definition $f(a)$

* A discontinuity at $x = 0$ or $x = 1$ is slightly different [compare the first footnote of § 2]. Under the present definition of the φ's it acts like an artificial discontinuity in the interior of the interval and has no effect on the sequence representing the function.

CONGRUENCES DETERMINED BY A GIVEN SURFACE.

By Claribel Kendall.

§ 1. Introduction.

It has been shown by Professor Wilczynski* that a non-developable analytic surface S may be regarded as an integrating surface of a non-involutory, completely integrable system of partial differential equations of the form

$$(1)\qquad \begin{aligned} y_{uu} + 2by_v + fy &= 0,\\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned}$$

where the subscripts denote partial differentiation, and where the coefficients, which are seminvariants, are analytic functions of u and v satisfying the integrability conditions

$$(2)\qquad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0,\\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0,\\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0. \end{aligned}$$

Then the curves $u =$ const., $v =$ const. form an asymptotic net on the surface S. We shall assume that in general $a' \neq 0$, $b \neq 0$, thus excluding ruled surfaces from our discussion.†

Under the above conditions (1) has exactly four linearly independent solutions

$$(3)\qquad y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

which are interpreted as the homogeneous coördinates of a point y on the surface S. The semicovariants of (1) are‡

$$(4)\qquad y, \quad y_u, \quad y_v, \quad y_{uv}.$$

Substituting the values (3) for y in (4) we obtain four points y, y_u, y_v, y_{uv} which are not coplanar since no relations of the form

$$\alpha y^{(k)} + \beta y_u^{(k)} + \gamma y_v^{(k)} + \delta y_{uv}^{(k)} = 0 \qquad (k = 1, 2, 3, 4)$$

can exist among them. For, otherwise (1) could have at most three linearly

* "Projective Differential Geometry of Curved Surfaces," first memoir, *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 246–7.

† Loc. cit., p. 260.

‡ "Projective Differential Geometry of Curved Surfaces," second memoir, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 79. We shall hereafter refer to this paper as Second memoir.

independent solutions. Hence these points may be used as the vertices of a local tetrahedron of reference for the purpose of studying S in the neighborhood of the point y. An expression of the form

$$\tau = a_1 y + a_2 y_u + a_3 y_v + a_4 y_{uv},$$

where a_1, a_2, a_3, a_4 are analytic functions of u and v, assumes four values $\tau^{(1)}$, $\tau^{(2)}$, $\tau^{(3)}$, $\tau^{(4)}$ corresponding to the four values of y. Hence τ determines a point whose local coördinates may be defined by writing

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = a_3, \qquad x_4 = a_4.$$

Consider the case when two such points

$$\begin{aligned} \tau_1 &= a_1 y + a_2 y_u + a_3 y_v + a_4 y_{uv}, \\ \tau_2 &= b_1 y + b_2 y_u + b_3 y_v + b_4 y_{uv} \end{aligned} \tag{5}$$

are given for every point y of the surface S. If we associate the line l, determined by τ_1 and τ_2, with each point of S, these lines form a congruence. Wilczynski and Green have considered such congruences in cases where the lines l pass through the point y or lie in the corresponding tangent plane. In this paper we shall consider the more general problems connected with the congruences determined by the lines l when l has an arbitrary position relative to S. General formulas will be obtained for the torsal curves and the guide curves (to be defined later), and for the focal points on l. These general formulas will then be applied to certain special congruences in connection with which various configurations of the lines themselves will be studied.

§ 2. Determination of the Developables of a Congruence.

For each point y of the surface S the points (5) determine a line l whose homogeneous line coördinates are

$$\omega_{ik} = a_i b_k - a_k b_i \qquad (i, k = 1, 2, 3, 4).$$

We wish to find the curves on S along which y must move in order that the corresponding line l of the congruence may describe a developable. These curves will be called the torsal curves of the surface S with respect to the congruence. Let u and v increase by amounts du and dv, where du and dv are infinitesimals, in such a way that the point y will change to $y + dy$, a new point on one of the torsal curves. Then the points τ_1 and τ_2 will move to $\tau_1 + d\tau_1$ and $\tau_2 + d\tau_2$ respectively. The line joining $\tau_1 + d\tau_1$ to $\tau_2 + d\tau_2$ is a generator of the developable consecutive to $\tau_1\tau_2$ and must intersect $\tau_1\tau_2$. Therefore τ_1, τ_2, $d\tau_1$, $d\tau_2$ must be coplanar. Now

$$d\tau_1 = (\tau_1)_u du + (\tau_1)_v dv, \qquad d\tau_2 = (\tau_2)_u du + (\tau_2)_v dv.$$

Hence, using (5) and (1),

$$
(6)\quad \begin{aligned}
d\tau_1 &= (A_1du + A_1'dv)y + (A_2du + A_2'dv)y_u \\
&\qquad + (A_3du + A_3'dv)y_v + (A_4du + A_4'dv)y_{uv}, \\
d\tau_2 &= (B_1du + B_1'dv)y + (B_2du + B_2'dv)y_u \\
&\qquad + (B_3du + B_3'dv)y_v + (B_4du + B_4'dv)y_{uv},
\end{aligned}
$$

where

$$
(7)\quad \begin{aligned}
A_1 &= (a_1)_u - fa_2 + (2bg - f_v)a_4, & A_1' &= (a_1)_v - ga_3 + (2a'f - g_u)a_4, \\
A_2 &= (a_2)_u + a_1 + 4a'ba_4, & A_2' &= (a_2)_v - 2a'a_3 - (g + 2a_u')a_4, \\
A_3 &= (a_3)_u - 2ba_2 - (f + 2b_v)a_4, & A_3' &= (a_3)_v + a_1 + 4a'ba_4, \\
A_4 &= (a_4)_u + a_3, & A_4' &= (a_4)_v + a_2, \\
B_1 &= (b_1)_u - fb_2 + (2bg - f_v)b_4, & B_1' &= (b_1)_v - gb_3 + (2a'f - g_u)b_4, \\
B_2 &= (b_2)_u + b_1 + 4a'bb_4, & B_2' &= (b_2)_v - 2a'b_3 - (g + 2a_u')b_4, \\
B_3 &= (b_3)_u - 2bb_2 - (f + 2b_v)b_4, & B_3' &= (b_3)_v + b_1 + 4a'bb_4, \\
B_4 &= (b_4)_u + b_3, & B_4' &= (b_4)_v + b_2.
\end{aligned}
$$

A necessary and sufficient condition that the points τ_1, τ_2, $d\tau_1$, $d\tau_2$ lie in a plane is that the determinant of the coördinates of the four points be zero. Expanding the determinant we obtain

$$
(8)\qquad Ldu^2 + 2Mdudv + Ndv^2 = 0,
$$

where

$$
\begin{aligned}
L &= \omega_{12}(A_3B_4) + \omega_{13}(A_4B_2) + \omega_{14}(A_2B_3) \\
&\qquad + \omega_{23}(A_1B_4) + \omega_{42}(A_1B_3) + \omega_{34}(A_1B_2), \\
2M &= \omega_{12}[(A_3B_4') + (A_3'B_4)] + \omega_{13}[(A_4B_2') + (A_4'B_2)] \\
&\qquad + \omega_{14}[(A_2B_3') + (A_2'B_3)] + \omega_{23}[(A_1B_4') + (A_1'B_4)] \\
&\qquad + \omega_{42}[(A_1B_3') + (A_1'B_3)] + \omega_{34}[(A_1B_2') + (A_1'B_2)], \\
N &= \omega_{12}(A_3'B_4') + \omega_{13}(A_4'B_2') + \omega_{14}(A_2'B_3') \\
&\qquad + \omega_{23}(A_1'B_4') + \omega_{42}(A_1'B_3') + \omega_{34}(A_1'B_2').
\end{aligned}
$$

Here (A_3B_4), etc., are the determinantal expressions $A_3B_4 - A_4B_3$, etc. We may then conclude

The torsal curves of the surface with respect to the congruence are determined by (8), *a quadratic differential equation which determines a net of curves on the surface S.*

§ 3. The Focal Points of the Lines l.

Each line l of the congruence belongs to two developables of the congruence, and the two points in which l touches the cuspidal edges of these developables (viz., the focal points of the line l) will now be found. Any point on the line l is given by an expression of the form

$$
(9)\qquad \varphi = \lambda\tau_1 + \mu\tau_2,
$$

where τ_1 and τ_2 are given by (5) and λ and μ are arbitrary functions of u and v. If φ is to be a focal point of l, then the tangent plane to the surface formed by all the points φ, as u and v vary, must contain the line l. Hence $\tau_1, \tau_2, \varphi_u, \varphi_v$ must be coplanar. Noting that

$$\varphi_u = \lambda_u\tau_1 + \mu_u\tau_2 + \lambda(\tau_1)_u + \mu(\tau_2)_u,$$
$$\varphi_v = \lambda_v\tau_1 + \mu_v\tau_2 + \lambda(\tau_1)_v + \mu(\tau_2)_v,$$

it follows that since τ_1 and τ_2 are coplanar with $\varphi_u - \lambda_u\tau_1 - \mu_u\tau_2$ and $\varphi_v - \lambda_v\tau_1 - \mu_v\tau_2$ they are also coplanar with $\lambda(\tau_1)_u + \mu(\tau_2)_u$ and $\lambda(\tau_1)_v + \mu(\tau_2)_v$. So that a necessary and sufficient condition that these points lie in the same plane is that the determinant of the coördinates of the points τ_1, τ_2, $\lambda(\tau_1)_u + \mu(\tau_2)_u$, $\lambda(\tau_1)_v + \mu(\tau_2)_v$ be zero. Expanding this determinant and using (7) we obtain

$$L'\lambda^2 + 2M'\lambda\mu + N'\mu^2 = 0, \tag{10}$$

where

$$\begin{aligned}
L' &= \omega_{12}(A_3A_4') + \omega_{13}(A_4A_2') + \omega_{14}(A_2A_3') \\
&\qquad + \omega_{23}(A_1A_4') + \omega_{42}(A_1A_3') + \omega_{34}(A_1A_2'), \\
2M' &= \omega_{12}[(A_3B_4') + (B_3A_4')] + \omega_{13}[(A_4B_2') + (B_4A_2')] \\
&\qquad + \omega_{14}[(A_2B_3') + (B_2A_3')] + \omega_{23}[(A_1B_4') + (B_1A_4')] \\
&\qquad + \omega_{42}[(A_1B_3') + (B_1A_3')] + \omega_{34}[(A_1B_2') + (B_1A_2')], \\
N' &= \omega_{12}(B_3B_4') + \omega_{13}(B_4B_2') + \omega_{14}(B_2B_3') \\
&\qquad + \omega_{23}(B_1B_4') + \omega_{42}(B_1B_3') + \omega_{34}(B_1B_2').
\end{aligned}$$

Here we have determinantal quantities similar to those in (8). The two values of λ/μ obtained from (10), substituted in (9), give the focal points of the line l, viz.,

$$\varphi_1 = \lambda_1\tau_1 + \mu_1\tau_2, \qquad \varphi_2 = \lambda_2\tau_1 + \mu_2\tau_2. \tag{11}$$

Their product determines a covariant

$$N'\tau_1^2 - 2M'\tau_1\tau_2 + L'\tau_2^2. \tag{12}$$

We may then conclude

The focal points of the lines l of the congruence are given by the factors of the covariant expression (12).

§ 4. The Guide Curves of the Congruence.

With each point y of the surface S is associated a unique plane through it, viz., the plane containing the line l determined by τ_1 and τ_2. This plane intersects the tangent plane at y in a well-determined line unless l lies in the tangent plane or passes through y. We shall now find the family of curves on S which will have these lines as tangents. These curves will

be called the *guide curves* of the surface with respect to the given congruence since the line l acts as a guide in determining the position of the plane. The equation of this plane is

$$\omega_{34}x_2 + \omega_{42}x_3 + \omega_{23}x_4 = 0. \tag{13}$$

It intersects the tangent plane $x_4 = 0$ in the line

$$\omega_{34}x_2 + \omega_{42}x_3 = 0, \qquad x_4 = 0.$$

If $u = u(t)$, $v = v(t)$ is the equation of the desired curve through y, the condition that this line be tangent to the curve gives

$$\omega_{34}du + \omega_{42}dv = 0 \tag{14}$$

as the differential equation of the required curves on S.

Miss Sperry* has developed the differential equation for the union curves on the surface S, which are curves such that the osculating plane of a point y on the curve will contain the generator of a congruence when this generator passes through the point y and does not lie in the tangent plane to S at y. We now desire to find the condition under which the guide curves are also union curves.

As before let the equation of a curve on S be given by $u = u(t)$, $v = v(t)$. The osculating plane of this curve at one of its points y is

$$2u'v'^2x_2 - 2u'^2v'x_3 - (u''v' - u'v'' + 2bu'^3 - 2a'v'^3)x_4 = 0, \tag{15}$$

where accents indicate differentiation as to t. If (13) and (15) are to represent the same plane,

$$\frac{2u'v'^2}{\omega_{34}} = \frac{-2u'^2v'}{\omega_{42}} = \frac{u'v'' - v'u'' - 2bu'^3 + 2a'v'^3}{\omega_{23}}. \tag{16}$$

$u' = 0$, $v' = 0$ is a solution of these equations but in this case the coordinates of the point y' would be (0, 0, 0, 0) which is not admissible. Assuming $u' \neq 0$ and introducing u as the parameter instead of t we obtain

$$\frac{dv}{du} = \frac{\omega_{34}}{\omega_{42}}, \qquad 2\omega_{23}\left(\frac{dv}{du}\right)^2 = \omega_{34}\left[\frac{d^2v}{du^2} - 2b + 2a'\left(\frac{dv}{du}\right)^3\right]. \tag{17}$$

Substituting the value of dv/du from the first equation of (17), which is the same as (14), into the second we find

$$\begin{aligned} 2\omega_{23}\omega_{34}\omega_{42} - \omega_{34}\omega_{42}(\omega_{42})_u + \omega_{42}^2(\omega_{34})_u \\ + \omega_{34}^2(\omega_{42})_v - \omega_{42}\omega_{34}(\omega_{34})_v + 2b\omega_{42}^3 + 2a'\omega_{44}^3 = 0 \end{aligned} \tag{18}$$

* "Properties of a Certain Projectively Defined Two-parameter Family of Curves on a General Surface," *American Journal of Mathematics*, Vol. XL (1918), pp. 213–224.

as the relation that must be satisfied by the line coördinates of l in order that the guide curves may be union curves.

Condition (18) is satisfied by $\omega_{34} = \omega_{42} = 0$. Two cases may arise depending on whether $\omega_{23} = 0$ or $\omega_{23} \neq 0$. If $\omega_{23} = 0$, we see from (16) that the solution is $u' = 0$, $v' = 0$, which has been excluded. Geometrically this occurs when the line l passes through the point y. As stated above this case has been considered by Miss Sperry from another point of view. If $\omega_{23} \neq 0$, the solution of (16) is $u' = 0$ if $v' \neq 0$ and is $v' = 0$ if $u' \neq 0$. This is the case when the line l lies in the tangent plane to S at y and does not pass through y.

We may then conclude

The one-parameter family of guide curves of the surface with respect to the given congruence has (14) *for its differential equation. In case* (18) *is satisfied the guide curves are also union curves. When l passes through the point y on the surface S or lies in the tangent plane to S at y, the guide curves are indeterminate.*

§ 5. The Osc-scroll-flec Congruences.

We shall now apply these results to some special congruences closely associated with a given surface. We begin by recalling what Wilczynski calls the osculating ruled surfaces of the first and second kinds, respectively. One of these, R_1, is the locus of the tangents to the asymptotic curves $v =$ const. along a fixed curve $u =$ const., and R_2 is the locus of the tangents to the asymptotic curves $u =$ const. along a fixed curve $v =$ const. The differential equations of R_1 and R_2 referred to our local tetrahedron of reference are given in Wilczynski's second memoir, pp. 81–82.

We shall have occasion to use the invariants of weights four, nine, and ten for R_1 and R_2. Expressed in terms of the coefficients of (1), for R_1 they are*

$$
\begin{aligned}
\theta_4 &= 2^6(a_u'^2 - 2a'a_{uu}' - 4a'^2 f - 4a'^2 b_v), \\
\theta_9 &= 4(C^3 + 8a'CC_{uv} - 8a'C_uC_v), \qquad (19)\\
\theta_{10} &= 2^6 a'^2 C_u^2 - C^2\theta_4,
\end{aligned}
$$

where†

$$C = 8a_{uv}' - 8\frac{a_u'a_v'}{a'} - 32a'^2 b. \qquad (20)$$

In obtaining θ_{10} and θ_9 use has been made of the relation

$$2a_v'\theta_4 - a'(\theta_4)_v = 16a'^2 C_u. \qquad (21)$$

The corresponding invariants θ_4', θ_9', θ_{10}' for R_2 are obtained from (19) by

* Second memoir, pp. 81, 84. θ_9 and θ_{10} were obtained by C. D. Meacham, a student at the University of Chicago.

† Loc. cit., p. 84.

the transpositions

$$(a'b), \quad (fg), \quad (uv), \quad (\theta_4\theta_4'), \quad (CC'), \tag{22}$$

where C' is obtained from C by the same transpositions.

The vanishing of θ_4 is the condition that R_1 have coincident branches to its flecnode curve, and $\theta_4' = 0$ is the corresponding condition for R_2. Consider for the present the case where $\theta_4 \neq 0$. Then R_1 may be referred to its flecnode curve. Let us denote the flecnodes on the generator yy_u of R_1 by η and ζ and let the lines ηr and ζs* be the flecnode tangents of R_1 at η and ζ respectively. To every point y of S there belongs such a line ηr and hence we obtain a congruence associated with S. Similarly ζs generates a second congruence. Relative to the osculating ruled surface R_2 two other congruences are determined in this way provided that the branches of the flecnode curve of R_2 do not coincide, i.e., provided $\theta_4' \neq 0$. We shall call these four congruences the *osc-scroll-flec congruences*, and we shall designate the congruences determined by ηr and ζs by Γ_1 and Γ_1' respectively and the corresponding congruences associated with R_2 by Γ_2 and Γ_2'. Referred to our local tetrahedron of reference, η, r, ζ, s are given by†

$$\begin{aligned} -32a'\sqrt{\theta_4}\,\eta &= (8a_u' - \sqrt{\theta_4})y - 16a'y_u, \\ -32a'\sqrt{\theta_4}\,r &= 64a'^2by + 2(8a_u' - \sqrt{\theta_4})y_v - 32a'y_{uv}, \\ 32a'\sqrt{\theta_4}\,\zeta &= (8a_u' + \sqrt{\theta_4})y - 16a'y_u, \\ 32a'\sqrt{\theta_4}\,s &= 64a'^2by + 2(8a_u' + \sqrt{\theta_4})y_v - 32a'y_{uv}. \end{aligned} \tag{23}$$

Hence, for the congruence Γ_1, the two points τ_1 and τ_2 of our general theory are given by

$$\begin{aligned} \tau_1 &= (8a_u' - \sqrt{\theta_4})y - 16a'y_u, \\ \tau_2 &= 64a'^2by + 2(8a_u' - \sqrt{\theta_4})y_v - 32a'y_{uv}. \end{aligned} \tag{24}$$

The corresponding formulas for the congruence Γ_1' differ from (24) merely in the sign of the radical.

Substituting (24) in (8) of § 2 and making use of (2), (19), (20), (21) we find that the torsal curves for Γ_1 are given by

$$\begin{aligned} [(\theta_4)_u^2 - 16a'^2\theta_4\theta_4']du^2 - 4(\theta_4)_u(8a'C_u + C\sqrt{\theta_4})dudv& \\ + 4(8a'C_u + C\sqrt{\theta_4})^2dv^2 &= 0. \end{aligned} \tag{25}$$

By changing the sign of the radical in (25) we obtain the torsal curves for

---

* Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces," Teubner, Leipzig, 1906, p. 124. We shall hereafter refer to this work as "Proj. Diff. Geom." The quantities r and s are there referred to as ρ and σ.

† Second memoir, p. 84.

Γ_1', viz.,

$$(26)\quad [(\theta_4)_u^2 - 16a'^2\theta_4\theta_4']du^2 - 4(\theta_4)_u(8a'C_u - C\sqrt{\theta_4})du dv + 4(8a'C_u - C\sqrt{\theta_4})^2dv^2 = 0.$$

The torsal curves for Γ_2 and Γ_2' may be obtained from (25) and (26) by means of the transpositions (22).

When $\theta_4 = 0$, R_1 has but one braneh to its flecnode curve. In that case the single flecnode tangent may be determined by*

$$2a'\zeta = a_u'y - 2a'y_u,$$
$$2a's = 8a'^2by + 2a_u'y_v - 4a'y_{uv},$$

where ζ is the flecnode. Γ_1 and Γ_1' coincide and the torsal curves for this congruence are found to be

$$(27)\qquad 4a'^2\theta_4'du^2 - C^2dv^2 = 0,$$

a conjugate net on S.

The focal points of the osc-scroll-flec congruences are found by factoring the covariant expression (12) of § 3. For Γ_1 when $\theta_4 \neq 0$, this covariant is

$$(28)\qquad (64b_v^2 - \theta_4')\tau_1^2 - 128bb_v\tau_1\tau_2 + 64b^2\tau_2^2,$$

aside from a factor $8a'C_u + C\sqrt{\theta_4}$ which is, in general, different from zero. Its vanishing makes $\theta_{10} = 0$. But $\theta_4 \neq 0$, $\theta_{10} = 0$ is the condition that R_1 have a straight line directrix.† An examination of the equations of the ruled surface R_1 when referred to its flecnode curves‡ shows that the flecnode curve C_η is also an asymptotic curve on R_1 and consequently is a straight line,§ the straight line directrix of R_1. Equation (25) shows that in this case the torsal net reduces to $du^2 = 0$. As y moves along $u =$ const., R_1 remains the same, the flecnode curve is a straight line and is, in fact, the line ηr; hence we see that the developable generated by ηr reduces to a straight line and consequently the congruence Γ_1 degenerates into a ruled surface.

For the congruence Γ_1' we obtain the same expression (28) for the focal points, but with the factor $8a'C_u - C\sqrt{\theta_4}$ omitted instead of $8a'C_u + C\sqrt{\theta_4}$. Its vanishing would cause θ_{10} to vanish and Γ_1' would degenerate into a ruled surface. If both of the factors of θ_{10} vanish, R_1 will have two straight line directrices and hence will belong to a linear congruence, and Γ_1 and Γ_1' will degenerate into ruled surfaces.

* Second memoir, p. 86.
† "Proj. Diff. Geom.," p. 167.
‡ Second memoir, pp. 83–84.
§ "Proj. Diff. Geom.," p. 150.

$= \frac{1}{2}[f(a-0)+f(a+0)]$; this follows from the evident possibility of expanding $f(x)$ in terms of the functions $f_0, f_1, f_2^{(1)}, \cdots$.

If the point a is dyadically irrational, $f(x)$ *cannot be expanded in terms of the* φ. The formal development of $f(x)$ converges in fact for every value of x other than a and diverges for $x = a$.* The convergence for $x \neq a$ follows, indeed, from Theorem IV. We proceed to demonstrate the divergence.

Use the dyadic notation

$$a = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots, \qquad a_n = 0 \text{ or } 1.$$

The partial sum

$$S_n^{(k)}(x) = \varphi_0(x)\int_0^1 f(y)\varphi_0(y)dy + \varphi_1(x)\int_0^1 f(y)\varphi_1(y)dy + \cdots + \varphi_n^{(k)}(x)\int_0^1 f(y)\varphi_n^{(k)}(y)dy$$

is in the sense of least squares the best approximation to $f(x)$ that can be formed from the functions $\varphi_0, \varphi_1, \cdots, \varphi_n^{(k)}$. It is therefore true that when $k = 2^{n-1}$, on every subinterval $\left(\frac{r}{2^n}, \frac{r+1}{2^n}\right)$ on which $f(x)$ is constant, $S_n^{(k)}(x)$ is also constant and equal to $f(x)$. On that subinterval $\left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$ which contains the point a, $S_n^{(k)}$ has the value

$$2^n a - m = \frac{a_{n+1}}{2^1} + \frac{a_{n+2}}{2^2} + \frac{a_{n+3}}{2^3} + \cdots, \tag{25}$$

which lies between zero and unity. Thus $S_n^{(k)}(x)$ $[n > 1]$ is a function with two points of discontinuity and which takes on three distinct values at its totality of points of continuity.

The infinite series corresponding to the sequence (25) is

$$\begin{aligned}\left(\frac{a_2}{2^1} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \cdots\right) + \left(\frac{a_3}{2^2} + \frac{a_4}{2^3} + \cdots - \frac{a_2}{2}\right)& \\ + \left(\frac{a_4}{2^2} + \frac{a_5}{2^3} + \frac{a_6}{2^4} + \cdots - \frac{a_3}{2}\right)& \\ + \left(\frac{a_5}{2^2} + \frac{a_6}{2^3} + \frac{a_7}{2^4} + \cdots - \frac{a_4}{2}\right) + \cdots&.\end{aligned} \tag{26}$$

Not all the numbers a_n after a certain point can be zero and not all of them

* This was pointed out for the set χ by Faber, *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. 19 (1910), pp. 104–112.

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can be unity, so the general term of the series (26) cannot approach zero and the sequence (25) cannot converge.

It is likewise true that the sequence (25) is not always summable and if summable may not be summable to the value $\frac{1}{2}$. Thus if we choose

$$a = \frac{1}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \frac{1}{2^7} + \cdots,$$

the sequence (25) is summable to the sum $\frac{2}{3}$. Likewise the sequence $S_n^{(k)}(x)$ for $x = a$ and where we consider all values of n and k, is summable to the value $\frac{2}{3}$.

The general behavior of $S_n^{(k)}(x)$ for $f(x)$ where we do not make the restriction $k = 2^{n-1}$ is quite easily found from the behavior for $k = 2^{n-1}$ and the relation

$$\varphi_n^{(i)}(a) \int_0^1 f(y)\varphi_n^{(i)}(y)dy = \varphi_n^{(k)}(a) \int_0^1 f(y)\varphi_n^{(k)}(y)dy,$$

which holds for all values of i, k, and n.

In fact there occurs a phenomenon quite analogous to Gibbs's phenomenon for Fornier's series. For the set φ, the approximating functions are uniformly bounded. The peaks of the approximating function $S_n^{(k)}$ disappear entirely for $k = 2^{n-1}$ but reappear (usually altered in height) for for larger values of n.

It is clear that the facts concerning the approximating curves for $f(x)$ hold without essential modification for a function of limited variation at a simple finite discontinuity, and that the facts for the summation of the approximating sequence hold without essential modification for a function continuous except at a simple finite discontinuity.

§ 7. The Uniqueness of Expansions.

We now study the possibility of a series of the form

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots \tag{27}$$

which converges on $0 \leqq x \leqq 1$ to the sum zero, with the possible exception of a certain number of points x. Faber has pointed out* that there exists a series of the functions $\chi_n^{(k)}(x)$ which converges to zero except at one single point, and the convergence is uniform except in the neighborhood of that point.

We state for reference the easily proved

LEMMA. *If the series* (27) *converges for even one dyadically irrational value of x, then* $\lim_{n=\infty} a_n = 0$.

* L. c., p. 111.

This lemma results immediately from the fact that $\varphi_n^{(k)}(x) = \pm 1$ if x is dyadically irrational.*

We shall now use this lemma to establish

THEOREM IX. *If the series* (27) *converges to the sum zero uniformly except in the neighborhood of a single value of x, then $a_n = 0$ for every n.*

We phrase the argument to apply when this exceptional value x_1 is dyadically irrational. If $x_1 > \frac{1}{2}$, we have for $0 \leqq x \leqq \frac{1}{2}$,

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots = 0,$$
$$(a_0 + a_1)\varphi_0(y) + (a_2 + a_3)\varphi_1(y) + (a_4 + a_5)\varphi_2(y) + \cdots = 0,$$

for every value of $y = 2x$. Then we have from the uniformity of the convergence,

$$a_0 + a_1 = 0, \qquad a_2 + a_3 = 0, \qquad a_4 + a_5 = 0, \qquad \cdots. \tag{28}$$

If $x_1 < \frac{3}{4}$, we have for $\frac{3}{4} \leqq x \leqq 1$,

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots = 0,$$

or for $0 \leqq y \leqq 1$, $y = 4x - 3$,

$$\begin{aligned}(a_0 - a_1 + a_2 - a_3)\varphi_0(y) &+ (a_4 - a_5 + a_6 - a_7)\varphi_1(y)\\ &+ (a_{4n} - a_{4n+1} + a_{4n+2} - a_{4n+3})\varphi_n(y) + \cdots = 0.\end{aligned}$$

From the uniformity of the convergence we have

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= 0,\\ a_4 - a_5 + a_6 - a_7 &= 0,\\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot, \end{aligned}$$

or from (28),

$$\begin{aligned} a_0 &= -a_1 = -a_2 = a_3,\\ a_4 &= -a_5 = -a_6 = a_7,\\ \cdot \quad &\cdot \quad \cdot \quad \cdot \quad \cdot. \end{aligned}$$

If $x_1 > \frac{5}{8}$, we have for $\frac{5}{8} \leqq x \leqq \frac{3}{4}$,

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots = 0,$$

or for $0 \leqq y \leqq 1$, $y = 8x - 5$,

$$\begin{aligned}(a_0 - a_1 - a_2 + a_3 - a_4 + a_5 + a_6 - a_7)\varphi_0(y)&\\ + (a_8 - a_9 - a_{10} + a_{11} - a_{12} + a_{13} + a_{14} - a_{15})\varphi_1(y) + \cdots &= 0.\end{aligned}$$

Then each of these coefficients must vanish, and hence

$$a_0 = -a_1 = -a_2 = a_3 = a_4 = -a_5 = -a_6 = a_7.$$

---

* This lemma is closely connected with a general theorem due to Osgood, *Transactions of the American Mathematical Society*, Vol. 10 (1909), pp. 337–346.

See also Plancherel, *Mathematische Annalen*, Vol. 68 (1909–1910), pp. 270–278.

Continuation in this way together with the Lemma shows that every a_n must vanish. This reasoning is typical and does not essentially depend on our numerical assumptions about x_1. Then Theorem IX is proved.

The reasoning is precisely similar if instead of the hypothesis of Theorem IX we admit the possibility of a finite number of points in the neighborhood of each of which the convergence is not assumed uniform:

Theorem X. *If the series*

$$a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) + \cdots$$

converges to the sum zero uniformly, $0 \leqq x \leqq 1$, *except in the neighborhood of a finite number of points, then* $0 = a_1 = a_2 = \cdots = a_n = \cdots$.

Harvard University,
May, 1922.

When $\theta_4 = 0$, the congruences Γ_1, Γ_1' coincide. The focal points are given by (28) aside from a factor C which is, in general, non-vanishing. If $C = 0$, the congruence degenerates into a ruled surface.

The focal points for the congruences Γ_2 and Γ_2', whether distinct or coincident, are obtained from (28) by means of the transpositions (22).

From the theorem of § 4 we find that the guide curves are given by $v =$ const. for Γ_1 and Γ_1', and by $u =$ const. for Γ_2 and Γ_2', as is obvious geometrically. The quantity $\omega_{34} = 0$ while $\omega_{42} \neq 0$, for Γ_1 and Γ_1', and $\omega_{42} = 0$, $\omega_{34} \neq 0$ for Γ_2 and Γ_2', hence the condition that the guide curves be union curves is not satisfied and the union curves do not exist.

In order to interpret geometrically special cases arising out of the discussion of the equations which have just been found, it will be convenient to give some geometric properties which follow when certain of the invariants are zero.

As previously mentioned, $\theta_4 = 0$ is the condition that R_1 may have coincident branches to its flecnode curve. It is also the condition that the focal sheets of the osc-scroll-flec congruences associated with R_2 coincide, as may be seen from the equation obtained from (28) by (22). In this case the cuspidal edges on this focal sheet are asymptotic curves on that surface. We also noted that $\theta_4 \neq 0$, $\theta_{10} = 0$ is the condition for R_1 to have a straight line directrix. Corresponding conditions hold when $\theta_4' = 0$ and when $\theta_4' \neq 0$, $\theta_{10}' = 0$. The conditions $C = 0$ and $C' = 0$ correspond to the cases when the asymptotic curves $v =$ const. and $u =$ const., respectively, belong to linear complexes.* If $C = C' = 0$, $\theta_4 \neq 0$, $\theta_4' \neq 0$, R_1 and R_2 belong to linear congruences with distinct directrices. If $C = \theta_4 = 0$, $C' = \theta_4' = 0$, these linear congruences have coincident directrices.†

By an examination of the discriminant of (25) for $\theta_4 \neq 0$, and of (27) for $\theta_4 = 0$, we find that the torsal curves for Γ_1 represent a one-parameter family of curves instead of a proper net in a number of special cases, which may be interpreted geometrically in the light of the foregoing properties. Among these special cases we find several where the asymptotic curves on S are torsal curves. When $u =$ const. is a torsal curve, Γ_1 degenerates into a ruled surface. When $v =$ const. is a torsal curve, the focal sheets of Γ_1 coincide. An examination of (25) shows that the asymptotic curves of S cannot both be torsal curves under the same conditions. Furthermore, the torsal curves form conjugate nets under special conditions which may be interpreted geometrically. Corresponding conditions hold for the torsal curves of Γ_1' and we can readily find the conditions under which these

* C. T. Sullivan, "Properties of Surfaces whose Asymptotic Curves belong to Linear Complexes," *Transactions of the American Mathematical Society*, Vol. 15 (1914), p. 178.

† Second memoir, p. 86.

curves coincide with the torsal curves of Γ_1'. Similar conditions hold relative to the torsal curves of the Γ_2 and Γ_2' congruences.

We shall now return to some general considerations regarding the osc-scroll-flec congruences. The tangents to the two torsal curves of Γ_1 at a point y of the surface are the lines joining y to the two points

$$y_u + (dv/du)_1 y_v, \qquad y_u + (dv/du)_2 y_v,$$

where $(dv/du)_1$ and $(dv/du)_2$ are the two roots of equation (25) regarded as a quadratic in dv/du. Let t_1 and t_2 be these two tangents. The directions conjugate to t_1 and t_2 are obtained by joining the point y to the points

$$y_u - (dv/du)_1 y_v, \qquad y_u - (dv/du)_2 y_v,$$

respectively. Using the term employed by Green* we shall call these two new tangents the reflected tangents of t_1 and t_2. The totality of these reflected tangents determines a new net on S which may, with Green, be called the reflected Γ_1-curves. They are determined by the differential equation

$$(29) \qquad [(\theta_4)_u^2 - 16a'^2\theta_4\theta_4']du^2 + 4(\theta_4)_u(8a'C_u + C\sqrt{\theta_4})du\,dv + 4(8a'C_u + C\sqrt{\theta_4})^2 dv^2 = 0,$$

an equation differing from (25) in the sign of the middle term only, since its roots are those of (25) with the signs changed. The torsal curves of Γ_1, the reflected Γ_1-curves, and the asymptotic curves at any point y of the surface S thus constitute three pairs in an involution. The Jacobian of the torsal curves of Γ_1 and the reflected Γ_1-curves gives the double elements of the involution, which constitute, of course, a pair of conjugate tangents, namely

$$(30) \qquad [(\theta_4)_u^2 - 16a'^2\theta_4\theta_4']du^2 - 4(8a'C_u + C\sqrt{\theta_4})^2 dv^2 = 0.$$

We may then say

The torsal curves of each osc-scroll-flec congruence, the corresponding reflected curves, and the asymptotic curves at any point y of the surface S constitute three pairs in an involution. The double elements of the involutions so determined give four unique projectively defined conjugate nets on S, one relative to each of the osc-scroll-flec congruences. Their differential equations are given by (30) *and the equations obtained from* (30) *by changing the sign of the radical and by the transpositions* (22) *applied to these two equations.*

The focal points of the line ηr for the Γ_1-congruence were found to be given by the expressions

$$\varphi_1 = \lambda_1\tau_1 + \mu_1\tau_2, \qquad \varphi_2 = \lambda_2\tau_1 + \mu_2\tau_2,$$

* Memoir on the general theory of surfaces and rectilinear congruences. *Transactions of the American Mathematical Society*, Vol. 20 (1919), p. 93.

where λ_1/μ_1 and λ_2/μ_2 are the roots of the quadratic equation

$$64b^2\lambda^2 + 128bb_v\lambda\mu + (64b_v^2 - \theta_4')\mu^2 = 0,$$

provided $\theta_{10} \neq 0$. A point which proves to be of considerable interest is obtained by finding the harmonic conjugate of η, i.e., τ_1, with respect to φ_1 and φ_2. It is given by

$$\alpha = 1/2(\lambda_1/\mu_1 + \lambda_2/\mu_2)\tau_1 + \tau_2.$$

From the above quadratic equation in λ/μ and from (24) this is found to be the point

$$\alpha = (-8a_u'b_v + 64a'^2b^2 + b_v\sqrt{\theta_4})y + 16a'b_vy_u + 2b(8a_u' - \sqrt{\theta_4})y_v - 32a'by_{uv}. \tag{31}$$

The corresponding point on ζs of the congruence Γ_1' is given by

$$\beta = (-8a_u'b_v + 64a'^2b^2 - b_v\sqrt{\theta_4})y + 16a'b_vy_u + 2b(8a_u' + \sqrt{\theta_4})y_v - 32a'by_{uv}. \tag{32}$$

Relative to Γ_2 and Γ_2' we obtain the two points α' and β', found from (31) and (32) by the transpositions (22).

The equation of the osculating quadric Q of the surface S at the point y is*

$$x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0. \tag{33}$$

It can readily be shown that *the lines $\alpha\beta$ and $\alpha'\beta'$ lie on this quadric and that they intersect one another at the point where the directrix of the second kind d'*† *intersects the osculating quadric.* $\alpha\beta$ intersects the side yy_u of the local tetrahedron of reference and $\alpha'\beta'$ intersects the side yy_v.

For each point y on S a line $\alpha\beta$ is determined even when $\theta_4 = 0$. If $\theta_4 = 0$, α and β coincide but the line $\alpha\beta$ is then the line joining $\alpha = \beta$ to the point where d' intersects the osculating quadric. Similarly for $\alpha'\beta'$. Consequently we have two new congruences associated with S. The torsal curves for these congruences are given by

$$\begin{aligned} 2^8b^2C'^2du^2 - 2^5b\theta_4'(C' + 32a'b^2)dudv + (\theta_4'^2 - 2^{10}b^4\theta_4)dv^2 &= 0, \\ (\theta_4^2 - 2^{10}a'^4\theta_4')du^2 - 2^5a'\theta_4(C + 32a'^2b)dudv + 2^8a'^2C^2dv^2 &= 0, \end{aligned} \tag{34}$$

respectively. If $\theta_4' = 0$, the torsal curves given in the first equation of (34) determine a conjugate net which coincides with the torsal net for the coincident Γ_2 and Γ_2' congruences. In this case the focal sheets of Γ_1 and Γ_1' coincide and $\alpha\beta$ becomes the line joining the coincident focal points on ηr and ζs. The line $\alpha'\beta'$ is now the line joining $\alpha' = \beta'$ to the point where d'

* Second memoir, p. 82.

† Loc. cit., p. 97.

intersects the osculating quadric. Similar conditions hold if $\theta_4 = 0$. If these conditions occur simultaneously, i.e., if $\theta_4' = \theta_4 = 0$, the torsal curves of the two congruences reduce to $du^2 = 0$ and to $dv^2 = 0$, respectively.

The focal points on $\alpha\beta$ and $\alpha'\beta'$ are given by the factors of the covariant expressions

$$
\begin{aligned}
&2^{10}b^2C'\beta^2 - 2^8a'b^2\theta_4'\alpha\beta - (a'\theta_4'^2 + 16b^2C'\theta_4)\alpha^2,\\
&2^{10}a'^2C\beta'^2 - 2^8a'^2b\theta_4\alpha'\beta' - (b\theta_4^2 + 16a'^2C\theta_4')\alpha'^2,
\end{aligned}
\tag{35}
$$

respectively. For $\theta_4' = 0$, the focal points on $\alpha\beta$ separate α and β harmonically. Similarly in the case $\theta_4 = 0$, α' and β' are separated harmonically by the focal points of the line $\alpha'\beta'$. When $\theta_4 = \theta_4' = 0$, the focal points of $\alpha\beta$ are coincident with β and the focal points of $\alpha'\beta'$ are coincident with β', i.e., the focal sheets of both of these congruences coincide.

The guide curves for $\alpha\beta$ and $\alpha'\beta'$ are found to be $u =$ const. and $v =$ const., respectively, as is obvious geometrically.

§ 6. The Congruences Determined by the Pairs of Complexes C_1, C' and C_2, C''.

Associated with a point y of the surface S there are four complexes,—the two complexes C_1 and C_2 which osculate the ruled surfaces R_1 and R_2 respectively and the two complexes C' and C'' which osculate the asymptotic curves of the first and second kinds respectively. Four of the pairs of complexes obtained from these are in involution, i.e., their bilinear invariants are zero.* Professor Wilczynski has considered† more in detail the congruences obtained from the directrices of the congruence common to the osculating complexes C' and C''. These he called the directrix congruences of the first and second kinds. We shall proceed to consider the other three pairs which are in involution. They are the pairs C_1, C'; C_2, C''; and C_1, C_2. The first two pairs may be considered together since they are symmetrically situated with respect to the surface S.

The equations of the complexes C_1 and C_2 referred to our local tetrahedron of reference are given by‡

$$
C_1 : a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{34}\omega_{34} + a_{42}\omega_{42} = 0,
\tag{36}
$$

where

$$
\begin{aligned}
&a_{12} = 0, \qquad a_{13} = 2^8a'^2C, \qquad a_{14} = a_{23} = 2^7a'(a'C_u + a_u'C),\\
&a_{34} = -2^9a'^3bC, \qquad a_{42} = -[C(\theta_4 + 64a_u'^2) + 2^7a'a_u'C_u],
\end{aligned}
$$

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* Second memoir, p. 95.

† Loc. cit., pp. 114–120.

‡ Loc. cit., pp. 85, 86, 89.

with the invariant

$$(37) \qquad A = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 2^8a'^2\theta_{10};$$

and

$$(38) \qquad C_2 : b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{34}\omega_{34} + b_{42}\omega_{42} = 0,$$

where

$$b_{12} = 2^8b^2C', \qquad b_{13} = 0, \qquad b_{14} = -b_{23} = 2^7b(bC'_v + b_vC'),$$
$$b_{34} = C'(\theta'_4 + 64b_v^2) + 2^7bb_vC'_v, \qquad b_{42} = 2^9a'b^3C',$$

with the invariant

$$(39) \qquad B = b_{12}b_{34} + b_{13}b_{42} + b_{14}b_{23} = -2^8b^2\theta'_{10}.$$

In writing down the above coefficients and invariants use has been made of (19) and (21). The equations of the complexes C' and C'' are*

$$(40) \qquad C' : -b_v\omega_{34} - b\omega_{14} + b\omega_{23} = 0,$$

with the invariant $A' = -b^2$,

$$(41) \qquad C'' : -a'_u\omega_{42} + a'\omega_{14} + a'\omega_{23} = 0,$$

with the invariant $A'' = a'^2$. It is to be noted that (38), (39) and (41) follow from (36), (37) and (40) respectively by applying the transpositions (22) and by interchanging the subscripts 2 and 3.

Consider the congruence determined by the linear complexes C_1 and C'. It will have two directrices, d_1 and d'_1, say. For every surface point we have two lines so determined. Hence d_1 and d'_1 will determine congruences associated with the surface S. We shall speak of these as the d_1-congruence and the d'_1-congruence. The equations of these directrices referred to the local tetrahedron of reference are found to be†

$$(42) \qquad \begin{aligned} a_{13}x_3 + (a_{14} + 16a'\sqrt{\theta_{10}})x_4 &= 0, \\ ba_{13}x_1 + b(a_{14} - 16a'\sqrt{\theta_{10}})x_2 - (ba_{34} + 16a'b_v\sqrt{\theta_{10}})x_4 &= 0, \end{aligned}$$

for d_1 and

$$(43) \qquad \begin{aligned} a_{13}x_3 + (a_{14} - 16a'\sqrt{\theta_{10}})x_4 &= 0, \\ ba_{13}x_1 + b(a_{14} + 16a'\sqrt{\theta_{10}})x_2 - (ba_{34} - 16a'b_v\sqrt{\theta_{10}})x_4 &= 0, \end{aligned}$$

for d'_1. In finding (42) and (43) it is useful to note that

$$a_{13}a_{24} - a_{14}^2 = 2^8a'^2\theta_{10}.$$

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* Second memoir, pp. 92, 94.

† In obtaining these equations use has been made of equations (67)–(69) on pp. 94–95 of Second memoir where (69) should read $A''\omega^2 - (A', A'')\omega + A' = 0$.

Since equations (43) differ from (42) only in the sign of the radical, we shall discuss the d_1-congruence in detail and obtain results for the other by changing the sign of $\sqrt{\theta_{10}}$.

As the two points (5) determining d_1 we may take the points of intersection of d_1 with the planes $x_2 = 0$ and $x_3 = 0$. d_1 intersects the edge yy_u of the local tetrahedron of reference, as must obviously be the case since yy_u is a line of both C_1 and C'. After substituting from (36) we have

$$(44)\quad \begin{aligned} \tau_1 &= (8a'C_u + 8a'_uC - \sqrt{\theta_{10}})y - 16a'Cy_u, \\ \tau_2 &= (32a'^2b^2C - b_v\sqrt{\theta_{10}})y \\ &\qquad + (8a'bC_u + 8ba'_uC + b\sqrt{\theta_{10}})y_v - 16a'bCy_{uv}. \end{aligned}$$

Let τ_1 and σ_1 be the points in which d_1 intersects the osculating quadric Q whose equation is given in (33). τ_1 is given in (44) and

$$(45)\quad \begin{aligned} \sigma_1 = [8(a'b_vC_u + a'_ub_vC - 8a'^2b^2C) + b_v\sqrt{\theta_{10}}]y - 16a'b_vCy_u \\ - 2b(8a'C_u + 8a'_uC + \sqrt{\theta_{10}})y_v + 32a'bCy_{uv}. \end{aligned}$$

Let τ'_1 and σ'_1 be the points in which d'_1 intersects Q. The coördinates of τ'_1 and σ'_1 are obtained from those of τ_1 and σ_1 respectively by changing the sign of $\sqrt{\theta_{10}}$. It can readily be shown that *the lines $\sigma_1\sigma'_1$, $\tau_1\sigma'_1$, and $\tau'_1\sigma_1$ lie on the quadric.* Moreover, *the line $\sigma_1\sigma'_1$ coincides with the line $\alpha\beta$ of* § 5. It can easily be verified that *τ_1 and τ'_1 are the complex points* on yy_u and hence are harmonically separated by η and ζ, the flecnodes on yy_u.* Then since ηr, ζs, $\tau_1\sigma'_1$ and $\tau'_1\sigma_1$ are all generators of the same kind on the osculating quadric, *σ_1 and σ'_1 are harmonically separated by α and β.* In § 5 we saw that $\alpha\beta$, i.e., *$\sigma_1\sigma'_1$, passes through the point where the directrix d' of the second kind intersects the osculating quadric.* Similar conditions hold relative to the lines determined in the same way from the complexes C_2, C''.

The torsal curves for the d_1-congruence are found, except for a non-vanishing factor $C/a'\theta_4\theta_{10}$, to be

$$(46)\qquad L_1du^2 + 2M_1dudv + N_1dv^2 = 0,$$

where

$$\begin{aligned} L_1 &= b^2C[C(\theta_4P + 32a'^2(\theta_4)_uC_u)^2 + 16a'^2C\theta_{10}(16a'^2\theta_4\theta'_4 - (\theta_4)^2_u) \\ &\qquad - 2^8a'^3C'\theta_4\theta_{10}\sqrt{\theta_{10}}], \\ 2M_1 &= 4a'b\theta_4[4a'C'\theta^2_{10} - bC\theta_9(\theta_4P + 32a'^2(\theta_4)_uC_u)], \\ N_1 &= a'^2\theta_4[4b^2\theta_4\theta^2_9 - \theta^2_{10}\theta'_4 - 16b^2C\theta_4\theta_{10}\sqrt{\theta_{10}}], \end{aligned}$$

in which

$$P = C\theta_4 - 2^6a'^2C_{uu} - 2^6a'a'_uC_u.$$

* "Proj. Diff. Geom.," p. 208.

The torsal curves for the d_1'-congruence are obtained from (46) by changing the sign of θ_{10}.

Since M_1 does not contain $\sqrt{\theta_{10}}$, we have by subtracting this equation from (46) and dividing by a non-vanishing factor

$$(47) \qquad 16a'C'du^2 + \theta_4 dv^2 = 0,$$

a conjugate net on S. *It is the only conjugate net of the involution determined by the torsal curves of the d_1- and d_2-congruences.* There is a corresponding conjugate net relative to C_2 and C''. At any point on S the four tangents determined by these two nets are harmonically related if $2^8a'bCC'+\theta_4\theta_4'=0$.

The focal points on d_1 are given by the factors of the covariant expression

$$(48) \qquad L_1'\tau_2^2 - 2M_1'\tau_1\tau_2 + N_1'\tau_1^2,$$

where

$$L_1' = 2^8CD, \qquad 2M_1' = 2^8C(b_vD + E),$$

$$N_1' = 2^7b_vCE + C\theta_4'(D + 8a'T\sqrt{\theta_{10}}) + 2^6b_v^2CD - 2^5a'bC'\theta_9(\theta_{10} + 8a'C_u\sqrt{\theta_{10}}),$$

in which

$$T = 8a'b\theta_9 + 2C_uP + \theta_uC^2, \qquad D = 2^8a'^3bC_u\theta_9 + \theta_{10}P - 4a'T\sqrt{\theta_{10}},$$

$$E = a'bC\theta_9(\theta_4)_u - 2a'^2C\theta_4'\theta_{10} + a'C'\theta_{10}\sqrt{\theta_{10}}.$$

The focal points on d_1' are obtained from (48) by changing the sign of $\sqrt{\theta_{10}}$.

The guide curves for the d_1- and d_1'-congruences are given by $v =$ const. and these curves are not union curves.

The torsal curves, the focal points, and the guide curves relative to the complexes C_2 and C'' may be obtained at once by means of the transpositions (22) together with the transposition $(\theta_{10}, \theta_{10}')$, (θ_9, θ_9').

§ 7. The Congruences Determined by the Pair of Complexes C_1 and C_2.

The equations of the linear complexes C_1 and C_2, whose bilinear invariant (A, B) is zero, are given in (36) and (38). The following relations

$$(49) \qquad a_{13}b_{24} = b_{12}a_{34}, \qquad a'^2\theta_{10}(b_{12}b_{34} - b_{14}^2) = b^2\theta_{10}'(a_{13}a_{24} - a_{14}^2),$$

between θ_{10} and θ_{10}' and the coefficients of (36) and (38) are useful in obtaining further results. As in the case of § 6 the equations of the two directrices of the congruence determined by C_1 and C_2 are given by

$$\begin{aligned} -\epsilon a'b_{12}\sqrt{\theta_{10}}x_2 + ba_{13}\sqrt{\theta_{10}'}x_3 + (ba_{14}\sqrt{\theta_{10}'} - \epsilon a'b_{14}\sqrt{\theta_{10}})x_4 &= 0, \\ (50) \quad \epsilon a'b_{12}\sqrt{\theta_{10}}x_1 + (ba_{14}\sqrt{\theta_{10}'} + \epsilon a'b_{14}\sqrt{\theta_{10}})x_3 \qquad & \\ + (ba_{24}\sqrt{\theta_{10}'} - \epsilon a'b_{24}\sqrt{\theta_{10}})x_4 &= 0, \end{aligned}$$

where $\epsilon = \pm 1$. Let δ_1 be the directrix determined by using $\epsilon = +1$ in

(50) and δ_2 the directrix corresponding to $\epsilon = -1$. It can be shown readily that *the points of intersection of δ_1 and δ_2 with the coördinate planes $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ are harmonic conjugates with respect to the osculating quadric Q.*

For the points τ_1 and τ_2 determining δ_1 we may take the points of intersection of δ_1 with the planes $x_2 = 0$ and $x_3 = 0$. The expressions for these points are

$$(51)\qquad \begin{aligned} \tau_1 = {} & (ba_{24}\sqrt{\theta'_{10}} - a'b_{24}\sqrt{\theta_{10}})y \\ & \qquad - (ba_{14}\sqrt{\theta'_{10}} - a'b_{14}\sqrt{\theta_{10}})y_u - a'b_{12}\sqrt{\theta_{10}}y_{uv}, \\ \tau_2 = {} & -(ba_{34}\sqrt{\theta'_{10}} - a'b_{34}\sqrt{\theta_{10}})y \\ & \qquad + (ba_{14}\sqrt{\theta'_{10}} - a'b_{14}\sqrt{\theta_{10}})y_v - ba_{13}\sqrt{\theta'_{10}}y_{uv}. \end{aligned}$$

An examination of the elements entering into the equation of the torsal curves of the congruence determined by δ_1 shows that we obtain the same equation over again if we apply the transformation (22) to it. In this sense we may speak of the torsal curves of the δ_1-congruence as symmetric. Similar conditions hold for the δ_2-congruence. These nets may be represented by

$$(52)\qquad \begin{aligned} & (\alpha_1\sqrt{\theta_{10}} + \alpha_2\sqrt{\theta'_{10}})du^2 + 2(\beta_1\sqrt{\theta_{10}} + \beta_2\sqrt{\theta'_{10}})dudv \\ & \qquad + (\gamma_1\sqrt{\theta_{10}} + \gamma_2\sqrt{\theta'_{10}})dv^2 = 0, \\ & (-\alpha_1\sqrt{\theta_{10}} + \alpha_2\sqrt{\theta'_{10}})du^2 + 2(-\beta_1\sqrt{\theta_{10}} + \beta_2\sqrt{\theta'_{10}})dudv \\ & \qquad + (-\gamma_1\sqrt{\theta_{10}} + \gamma_2\sqrt{\theta'_{10}})dv^2 = 0, \end{aligned}$$

where γ_1, γ_2, β_2 are obtained from α_2, α_1, β_1 respectively by (22).

If $\theta_{10} = 0$, $\theta'_{10} \neq 0$, the invariant of the complex C_1 is zero and hence C_1 is a special linear complex unless $\theta_{10} = 0$ by virtue of C being zero in which case the complex C_1 is indeterminate. Excluding that case we see that the congruence determined by C_1 and C_2 has two coincident straight line directrices.* This directrix is given by the two points whose coördinates are

$$(53)\qquad (a_{24}, -a_{14}, 0, 0), \quad (a_{34}, 0, -a_{14}, a_{13}).$$

It is easy to verify that this line lies on the osculating quadric since $\theta_{10} = 0$. In the case we are considering we have but one congruence associated with the surface S instead of two. Its torsal curves may be obtained from (8) without a great deal of algebraic work. It is obvious geometrically that its guide curves are $v =$ const. since the directrix intersects the line yy_u.

If $\theta'_{10} = 0$, $C' \neq 0$, $\theta_{10} \neq 0$, we obtain a congruence generated by the line determined by the points whose coördinates are

$$(54)\qquad (b_{34}, 0, -b_{14}, 0), \quad (b_{24}, -b_{14}, 0, b_{12}).$$

* Second memoir, p. 163.

This line lies on the osculating quadric since $\theta'_{10} = 0$ and it intersects the line yy_v. Consequently the guide curves of the congruence determined by it are $u =$ const.

If $\theta_{10} = 0$, $\theta'_{10} = 0$, $C \neq 0$, $C' \neq 0$, the two complexes C_1 and C_2 are both special and the congruence determined by them degenerates into two systems of ∞^2 lines, viz., all the lines in the plane of the two axes of the special complexes and all the lines through their point of intersection.* This point of intersection is the point in which the lines (53) and (54) meet since in this case (53) and (54) are the two axes which are generators of different kinds on the osculating quadric.

Returning to the general case we find that the guide curves of the δ_1-congruence are given by

$$ba_{13}\sqrt{\theta'_{10}}du - a'b_{12}\sqrt{\theta_{10}}dv = 0, \tag{55}$$

and of the δ_2-congruence by

$$ba_{13}\sqrt{\theta'_{10}}du + a'b_{12}\sqrt{\theta_{10}}dv = 0. \tag{56}$$

Hence *the guide curves of the surface S with respect to these two congruences together constitute a projectively defined conjugate net on S, namely, the net*

$$a'^2C^2\theta'_{10}du^2 - b^2C'^2\theta_{10}dv^2 = 0. \tag{57}$$

* Loc. cit., p. 163.

LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH A CONTINUOUS INFINITUDE OF VARIABLES.*

BY I. A. BARNETT.

The purpose of this paper is to extend the theory of the linear partial differential equations

$$(1) \qquad \frac{\partial F}{\partial \tau}(\tau; u_1, \cdots, u_n) + \sum_{i=1}^{n} f_i(\tau; u_1, \cdots, u_n) \frac{\partial F}{\partial u_i} = 0,$$

$$(2) \qquad \sum_{i=1}^{n} f_i(u_1, \cdots, u_n) \frac{\partial F}{\partial u_i}(u_1, \cdots, u_n) = 0,$$

to equations which involve a continuous infinitude of variables. Since both of these equations involve the known functions $f_1, \cdots, f_n$ linearly, this suggests immediately the use of the Stieltjes integral for expressing the equations in the transcendental case. The equations to be studied are of the form

$$(1') \qquad \frac{\partial F}{\partial \tau}[\tau, u] + \int_0^1 f[\xi, \tau, u] d_\xi \varphi[\xi, \tau, u] = 0$$

and

$$(2') \qquad \int_0^1 f[\xi, u] d_\xi b[\xi, u] = 0,$$

where ξ is a real variable, u is a continuous function of the variable ξ', f a given functional operation, F the functional sought, and φ and b stand for certain associated functionals of F. All of these symbols will be defined more precisely in § 1.

Equations similar to (1′) and (2′) but involving derivatives of functions of lines have already been studied by Volterra who considers the equation

$$(3) \qquad \frac{\partial F[\tau, u]}{\partial \tau} + \int_0^1 f[\xi, \tau, u] F'[\xi, \tau, u] d\xi = 0,$$

where F' denotes the Volterra derivative of the functional F with respect to u.†

* Presented to the Society in two papers, (1) at Chicago, December, 1918, and (2) at New York, April, 1920.

† Volterra, "Equazioni integro-differentiali ed equazioni alle derivate funzionali," *Atti della Reale Accademia dei Lincei* (1914), Vol. XXIII, serie 6, 1st semester, p. 551.

In § 1 some preliminary matters concerning differential equations involving functionals, implicit functional equations, and Fréchet differentials will be stated which will be found useful in the sequel. In § 2 solutions of (1′) will be shown to exist. This will come out as an application proved in a previous paper.* In § 3 there will be considered the question of finding all the solutions of a particular type. This will necessitate the use of an implicit functional equation studied by Lamson. Finally, in the last section analogous questions will be considered for the homogeneous equation (2′).

§ 1. Some Preliminary Lemmas.

The notations and definitions of the following lemma will be found in *Diff. Eq.* It is a condensation of Theorems I, III and IV.

LEMMA 1. *If the functional* $f[\xi, \tau, u]$ *is such that it possesses in the set*

$$(A_0) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \alpha, \qquad ||u - u_{00}|| \leqq \alpha$$

a difference function A, *satisfying*

$$f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u] = A[\xi, \eta, u, \bar{\eta}, \bar{u}; \bar{\eta} - \eta, \bar{u} - u],$$

which difference function besides having the linearity and modular properties designated by (2) *and* (3) [*Diff. Eq.*, § 2] *has also the modified continuity property* (1′); *then there exists a set of elements* B_0†

$$(B_0) \quad 0 \leqq \xi \leqq 1, \quad |\tau - \tau_0| \leqq \beta, \quad |\tau_0 - \tau_{00}| \leqq \beta, \quad ||u_0 - u_{00}|| \leqq \beta,$$

in which the unique solution $u = v[\xi, \tau, \tau_0, u_0]$ *of the functional equation*

$$(4) \qquad \frac{\partial u}{\partial \tau}(\xi, \tau) = f[\xi, \tau, u]$$

reducing to $u = u_0$ *for* $\tau = \tau_0$ *is defined and continuous with respect to all of its arguments and possesses a difference function* $B[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$.

One may simplify the statement of Lemma 2 by the following

Definition. The difference function $\Gamma[\xi, \tau, u, \bar{u}; \tilde{u}]$ is said to have a *reciprocal* for $\tau = \tau_0$, $u = \bar{u} = u_0$ if there exists a functional $\bar{\Gamma}[\xi, \tau, u, \bar{u}; \tilde{u}]$ with the properties (1), (2) and (3) of a difference function (*Diff. Eq.*, § 2) such that

$$\bar{\Gamma}[\xi, \tau_0, u_0, u_0; \Gamma[\xi, \tau_0, u_0, u_0; \tilde{u}]] = \tilde{u}(\xi)$$

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* "Differential Equations Involving a 'Continuous Infinitude of Variables," this JOURNAL, Vol. XLIV, p. 172. This paper will be referred to as *Diff. Eq.*

† The set (B_0) is in a form somewhat different from that used in *Diff. Eq.* (§ 3) but it can always be taken in the form given here which for the purposes of the present paper is more convenient.

and with the further property that the vanishing of $\Gamma[\xi, \tau_0, u_0, u_0; \tilde{u}]$ identically in ξ implies that $\tilde{u}(\xi) = 0$ identically in ξ.

If now in Lamson's theorem* one interprets the range P to be the variable ξ, the class $\mathfrak{M}$ as the class of all continuous functions defined for the interval $0 \leqq \xi \leqq 1$, and the modulus $||u(\xi)||$ to be the maximum of the function u, the following lemma results:

LEMMA 2. *If, in the implicit functional equation*

$$G[\xi, \tau, u] = z(\xi), \tag{5}$$

the given functional G has the properties

(1) *Equation (5) is satisfied by the element* (τ_0, u_0, z_0),

(2) *G is real, single valued and continuous in its arguments,*

(3) *It has a difference function $\Gamma[\xi, \tau, u, \bar{u}; \tilde{u}]$ for all $(\xi, \tau, u, \bar{u})$ in the set*

$$0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \sigma, \qquad ||u - u_0|| \leqq \sigma, \qquad ||\bar{u} - u_0|| \leqq \sigma,$$

(4) *For $\tau = \tau_0$, $u = \bar{u} = u_0$, Γ has a reciprocal $\bar{\Gamma}$;*
then there exists a constant $\sigma_1 \leqq \sigma$ such that the equation (5) has one and only one solution

$$u = H[\xi, \tau, z].$$

The functional H is uniformly continuous in its arguments and reduces to u_0 for $z = z_0$.

The lemma as stated is really not a special instance of Lamson's result since the left-hand side of equation (5) contains a parameter τ but with the hypotheses of Lemma 2 one could carry through step by step the proof given by Lamson and show that the conclusions of the lemma result.

Remark.—One could prove that the solution $H[\xi, \tau, z]$ has a difference function in its domain of definition. This proof could be effected by using Lemma 4, *Diff. Eq.* (§ 2), and a method of proof similar to Theorem IV of the same paper.

Suppose now that $u(\xi; \alpha)$ is for each fixed α of the interval $0 \leqq \alpha \leqq 1$ a continuous function of ξ and for each fixed ξ of $0 \leqq \xi \leqq 1$ a differentiable function of α. Suppose also that the functional $F[\xi, \tau, u]$ has a difference function $\Phi[\xi, \tau, u, \bar{\tau}, \bar{u}; \tilde{\tau}, \tilde{u}]$ so that

$$F[\xi, \bar{\tau}, \bar{u}] - F[\xi, \tau, u] = \Phi[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau} - \tau, \bar{u} - u],$$

where $u = u(\xi'; \alpha)$, $\bar{u} = u(\xi'; \bar{\alpha})$ and Φ has the properties (1), (2) and (3). (*Diff. Eq.*, § 2.)

* K. Lamson, "A General Implicit Function Theorem," this JOURNAL, Vol. XLII, pp. 243–256.

Definition.—$\Phi[\xi, \tau, u, \tau, u; \bar{\tau}, \tilde{u}]$ is called the differential of $F[\xi, \tau, u]$ at the element (τ, u) with the argument element $(\bar{\tau}, \tilde{u})$.

Lemma 3. *If a functional $F[\xi, \tau, u]$ has a difference function Φ, then $d/d\alpha\, F[\xi, \tau, u(\xi, \alpha)]$ is the differential at the element (ξ, τ, u) with the argument function $(0, \partial u/\partial\alpha)$.*

By hypothesis

$$F[\xi, \tau, u + \Delta u] - F[\xi, \tau, u] \equiv F[\xi, \tau, u(\xi', \alpha + \Delta\alpha)] - F[\xi, \tau, u(\xi', \alpha)]$$
$$= \Phi[\xi, \tau, u, \tau, u + \Delta u; 0, \Delta u]$$

and hence by the linearity property of the difference function

$$\frac{F[\xi, \tau, u + \Delta u] - F[\xi, \tau, u]}{\Delta\alpha} = \Phi\left[\xi, \tau, u, \tau, u + \Delta u; 0, \frac{\Delta u}{\Delta\alpha}\right].$$

Thus by the continuity property of Φ it follows that as $\Delta\alpha \to 0$, the right-hand member tends to the expression $\Phi[\xi, \tau, u, \tau, u; 0, u_\alpha]$ as desired.

It would follow from Riesz's representation of a linear functional that the preceding lemma could also be written in the form

$$\frac{d}{d\alpha} F[\xi, \tau, u] = \int_0^1 \frac{\partial u(\xi', \alpha)}{\partial\alpha}\, d_{\xi'}\varphi[\xi, \xi', \tau, u]$$

and as Fréchet has pointed out there could be but one representation if it is specified that φ is identically zero for $\xi' = 1$ and that its discontinuities in ξ' are regular. This will always be supposed in what follows. The functional will be called the functional *associated* with the differential Φ of F or, more simply, the functional associated with F.

Corollary.—If the function $u(\xi, \alpha)$ is taken to be $u(\xi) + \alpha\tilde{u}(\xi)$, then

$$\frac{d}{d\alpha} F[\xi, \tau, u + \alpha\tilde{u}]\big|_{\alpha=0} = \int_0^1 \tilde{u}(\xi')d_{\xi'}\varphi[\xi, \xi', \tau, u].$$

§ 2. Existence of Solutions of the Non-homogeneous Equation.

Consider the equation

$$(1') \qquad \frac{\partial F}{\partial\tau}[\tau, u] + \int_0^1 f[\xi, \tau, u]d_\xi\varphi[\xi, \tau, u] = 0,$$

where φ is the functional associated with F. One can now prove the following theorem.

Theorem 1. *Suppose $v[\xi, \tau, \tau_0, u_0]$ is the unique solution of the functional equation*

$$\frac{\partial}{\partial\tau} u(\xi, \tau) = f[\xi, \tau, u],$$

of the kind described in Lemma 1, then for each fixed τ' in $|\tau' - \tau_{00}| \leqq \beta$, the functional

$$F[\xi, \tau, u] \equiv v[\xi, \tau', \tau, u] \tag{6}$$

is defined and continuous in the set of elements

$$0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \gamma, \qquad ||u - u_{00}|| \leqq \gamma,$$

where γ is some positive number. Furthermore, F has a difference function $\Phi[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau}, \tilde{u}]$ and satisfies equation (1′).

Suppose

$$u = v[\xi, \tau, \tau', u']$$

is the solution of equation (4) passing through the initial element (τ', u'). Then it follows immediately from the uniqueness of the solution that

$$u' = v[\xi, \tau', \tau, u] = v[\xi, \tau', \tau, v[\xi, \tau, \tau', u']]. \tag{7}$$

Now, by Lemma 1, $v[\xi, \tau', \tau, u]$ possesses a difference function with respect to the arguments τ and u. It follows readily from this that $\partial v/\partial\tau$ exists. Hence, differentiating both sides of (7) with respect to τ, and making use of Lemma 3, one finds that

$$0 = \frac{\partial v}{\partial \tau}[\xi, \tau', \tau, u] + \bar{\Phi}\left[\xi, \tau, v, v; 0, \frac{\partial v}{\partial \tau}\right],$$

or

$$0 = \frac{\partial v}{\partial \tau}[\xi, \tau', \tau, u] + \int_0^1 \frac{\partial v}{\partial \tau}[\xi', \tau', \tau, u] d_{\xi'}\bar{\varphi}[\xi, \xi', \tau', \tau, u],$$

where $\bar{\varphi}$ is the functional associated with $v[\xi, \tau', \tau, u]$ when all but the last argument are kept fixed. But in view of (4) and (6) the last relation may be written

$$0 = \frac{\partial F}{\partial \tau}[\xi, \tau, u] + \int_0^1 f[\xi', \tau, u] d_{\xi'}\varphi[\xi, \xi', \tau, u],$$

where φ is the functional associated with F.

Corollary.—If the associated functional $\varphi[\xi, \xi', \tau, u]$ has a continuous derivative $\varphi_{\xi'}[\xi, \xi', \tau, u] \equiv \partial\varphi/\partial\xi'$, for all points of the interval $0 \leqq \xi' \leqq 1$, then there exists a solution of the equation

$$\frac{\partial F}{\partial \tau}[\xi, \tau, u] + \int_0^1 f[\xi', \tau, u] \frac{\partial \varphi}{\partial \xi'}[\xi, \xi', \tau, u] = 0.$$

This is in essence Volterra's result already cited, for the functional $\partial\varphi/\partial\xi'$

could be readily identified with the Volterra derivative of F with respect to u.

Example.—Consider the equation

$$\frac{\partial F}{\partial \tau}[\tau, u] + \int_0^1 \left\{ \int_0^1 K(\eta, \zeta)u(\zeta)d\zeta \right\} d_\eta \varphi[\eta, \tau, u] = 0, \tag{8}$$

where $K(\eta, \zeta)$ is symmetric. Here the functional $f[\xi, \tau, u]$ of equation (1′) is given by

$$f[\xi, \tau, u] = \int_0^1 K(\xi, \zeta)u(\zeta)d\zeta,$$

and the corresponding equation (4) is

$$\frac{\partial u}{\partial \tau}(\xi, \tau) = \int_0^1 K(\xi, \zeta)u(\zeta)d\zeta,$$

with the initial condition

$$u(\xi, \tau_0) = u_0(\xi).$$

Now it has been shown* that the unique solution of this system is given by

$$v[\xi, \tau] = u_0(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)(e^{(\tau-\tau_0)/\lambda_i} - 1) \int_0^1 \varphi_i(\zeta)u_0(\zeta)d\zeta,$$

where the λ_i are the characteristic numbers of K and the φ_i are the corresponding normed orthogonal characteristic functions. It follows therefore by the theory just developed that

$$F[\xi, \tau, u] \equiv u(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)(e^{-(\tau-\tau_0)/\lambda_i} - 1) \int_0^1 \varphi_i(\zeta)u(\zeta)d\zeta$$

is for every ξ of the interval (0, 1) a solution of equation (8). This could be readily verified directly.

§ 3. The General Solution of Equation (1′).

It is desired in this section to obtain a solution of equation (1′) in terms of which all others of a certain type are expressible. The following theorem is first proved.

THEOREM 2. *If $F[\xi, \tau, u]$ is for every ξ of (0, 1) a solution of equation (1′) of the kind described in Theorem 1, then*

$$G[\tau, u] \equiv L[F[\xi, \tau, u]] \tag{9}$$

* See paper by writer, "Integro-differential Equations with the Constant Limits of Integration," *Bull. of Amer. Math. Soc.*, Vol. XXVI, pp. 193–203.

where L is an arbitrary functional eliminating the argument ξ and possessing for each fixed element (τ, u) of the set defined by the inequalities

$$|\tau - \tau_{00}| \leqq \beta, \qquad ||u - u_{00}|| \leqq \beta, \tag{10}$$

a difference function $\Lambda[F, \bar{F}; \bar{\bar{F}}]$, is also a solution of equation (1′) *defined and continuous in* (10).

In the first place, one could readily verify that G has a difference function with respect to τ and u. This follows from the fact that both F and L possess difference functions with respect to their arguments. Hence it follows by the Corollary to Lemma 3 that

$$\frac{dG}{d\alpha}[\tau + \alpha, u + \alpha\bar{u}]|_{\alpha=0} = \frac{\partial G}{\partial \tau} + \int_0^1 \bar{u}(\xi) d_\xi \gamma[\xi, \tau, u],$$

where γ is the functional associated with G when the argument τ is kept fixed. But by (9) and Lemma 3 the left member of the preceding relation may be written

$$\frac{d}{d\alpha} L[F[\xi, \tau + \alpha, u + \alpha\bar{u}]]|_{\alpha=0} = \int_0^1 \frac{\partial F}{\partial \alpha}[\xi, \eta, \tau + \alpha, u + \alpha\bar{u}]|_{\alpha=0} d_\eta \lambda(\eta, F),$$

where λ is the functional associated with the differential $\Lambda[F, \bar{F}, \bar{\bar{F}}]$ of L. Applying again the Corollary of Lemma 3 to the last expression, one obtains finally

$$\begin{aligned}\frac{\partial}{\partial \tau} G[\tau, u] &+ \int_0^1 \bar{u}(\xi) d_\xi \gamma[\xi, \tau, u] \\ &= \int_0^1 \left\{ \frac{\partial F}{\partial \tau}[\eta, \tau, u] + \int_0^1 \bar{u}(\xi) d_\xi \varphi[\xi, \eta, \tau, u] \right\} d_\eta \lambda(\eta, F).\end{aligned}$$

Substituting now $f[\xi, \tau, u]$ for $\bar{u}(\xi)$ and remembering that F is a solution of equation (1′), one sees that the right side of the last equation vanishes identically in τ and u, proving that

$$\frac{\partial G}{\partial \tau}[\tau, u] + \int_0^1 f[\xi, \tau, u] d_\xi \gamma[\xi, \tau, u] = 0$$

as desired.

On the basis of the implicit function theorem proved in Lemma 2 one is now able to give a method for expressing all solutions of a certain type. This is embodied in the following theorem:

THEOREM 3. *If $F[\xi, \tau, u]$ is a solution of the kind described in Theorem* 1, *then any solution $L[\tau, u]$ of equation* (1′) *possessing a difference function $\Lambda[\tau, u, \bar{\tau}, \bar{u}; \bar{\bar{\tau}}, \bar{\bar{u}}]$ in the set of elements defined by* (10) *is a continuous functional of F when in F the element (τ, u) is thought of as fixed.*

Consider the functional equation

$$F[\xi, \tau, u] = z(\xi). \tag{11}$$

It is desired to show that this equation satisfies all the hypotheses of Lemma 2 and hence can be solved for u. In the first place it is clear from the definition (6) of F that equation (1′) is satisfied by $\tau = \tau_0$, $u = u_0$, $z = u_0$. Furthermore, since $F[\xi, \tau_0, u] \equiv u(\xi)$, it follows that the difference function of F has a reciprocal for $\tau = \tau_0$ and $u = \bar{u} = u_0$. Finally, F has all the required continuity properties. Hence, Lemma 2 is applicable and one may conclude that

$$u(\xi) = H[\xi, \tau, z], \tag{12}$$

where H is a continuous functional of its arguments for $0 \leqq \xi \leqq 1$, and for τ, z in a suitable neighborhood of $\tau = \tau_0$, $z = u_0$.

Substituting (12) in $L[\tau, u]$, one obtains

$$L[\tau, H[\xi, \tau, z]] \equiv M[\tau, z]$$

and it remains to show, in order to prove the theorem, that M does not involve τ explicitly. Proceeding as in the proof of the last theorem, one may obtain the relation

$$\begin{aligned}\frac{\partial L}{\partial \tau}[\tau, u] + \int_0^1 f[\eta, \tau, u] d_\eta \lambda[\eta, \tau, u] \\ = \frac{\partial M}{\partial \tau} + \int_0^1 \left\{ \frac{\partial F}{\partial \tau} + \int_0^1 f[\xi, \tau, u] d_\xi \varphi[\xi, \eta, \tau, u] \right\} d_\eta \mu[\eta, F],\end{aligned}$$

where μ is the functional associated with the differential of $M[\tau, u]$ which can be shown to exist since both L and H have difference functions (see Remark, Lemma 2). But L and F are by hypothesis solutions of equation (1′), so that

$$\frac{\partial M}{\partial \tau} = 0$$

as desired.

§ 4. The Homogeneous Equation.

Consider now the homogeneous equation

$$\int_0^1 f[\xi, u] d_\xi \sigma[\xi, u] = 0, \tag{2′}$$

where σ is the functional associated with the differential $\sum$ of the unknown functional $S[\xi, u]$.

Let $F[\xi, \tau, u]$ be a solution of equation (1′) of the kind described in Theorem 1. Consider an arbitrary functional $G[F[\xi, \tau, u]]$ of F. In other words, when the arguments τ, u are kept fixed in F, then for every F whose modulus lies between suitable limits, G yields a real number. This arbitrary functional will then depend upon τ and u and will be denoted by $H[\tau, u]$. Let it be assumed that $H[\tau, u]$ has the following properties:

(1) There exists an element (τ_0, u_0) for which

$$H[\tau_0, u_0] = 0.$$

(2) $H[\tau, u]$ as well as $H_\tau[\tau, u]$ are continuous functionals of their arguments for

$$|\tau - \tau_0| \leqq \epsilon, \qquad ||u - u_0|| \leqq \epsilon,$$

where ϵ is some positive number.

(3) $\left.\dfrac{\partial H}{\partial \tau}\right|_{\substack{\tau=\tau_0\\ u=u_0}} \neq 0.$

(4) $H[\tau, u]$ has a difference function in $|\tau - \tau_0| \leqq \epsilon$, $|\bar{\tau} - \tau_0| \leqq \epsilon$, $||u - u_0|| \leqq \epsilon$, $||\bar{u} - u_0|| \leqq \epsilon$.

It can be readily shown that such functionals H exist. For, it is clear from the defintion of $F[\xi, \tau, u]$ that

$$\left.\frac{\partial F}{\partial \tau}[\xi, \tau, u]\right|_{\substack{\tau=\tau_0\\ u=u_0}} = -u_0(\xi)$$

and excluding the case for which $u_0(\xi) \equiv 0$, it follows that if one takes

$$H[\tau, u] \equiv F[\xi_1, \tau, u] - u_0(\xi_1),$$

where ξ_1 is a value of ξ for which $u_0(\xi) \neq 0$, then the functional H will satisfy the conditions (1), (2), (3), (4).

Consider now the functional equation

$$H[\tau, u] = 0, \tag{13}$$

where $H \equiv G[F[\xi, \tau, u]]$ and has the properties (1), (2), (3), (4). If one applies to equation (13) a theorem proved by Bliss,* one may show that there exists a unique solution

$$\tau = K[u], \tag{14}$$

where K is a continuous functional of u in a neighborhood $||u - u_0|| \leqq \delta < \epsilon$ reducing for $u = u_0$ to $\tau = \tau_0$. It could be shown furthermore

* Bliss, *Transactions of the American Mathematical Society*, Vol. 21, April, 1920, p. 90.

that K has a difference function. The associated functional of the functional K will be denoted by $\kappa[\xi u]$. Substituting (14) in $F[\xi, \tau, u]$, one obtains

$$F[\xi, K[u], u] \equiv S[\xi, u] \tag{15}$$

and it is desired to prove that S is a solution of the equation (2′).

In the first place it follows that since both K and F have difference functions with respect to their arguments, S has a difference function with respect to u. Let σ be the associated functional of the differential $\sum$ of S. By Corollary to Lemma 3 one obtains from (15) the relation

$$\int_0^1 \bar{u}(\xi) d_\xi \sigma[\xi, u] = \frac{\partial F}{\partial \tau} \int_0^1 \bar{u}(\xi) d_\xi \kappa[\xi, u] + \int_0^1 \bar{u}(\xi) d_\xi \varphi[\xi, K, u]. \tag{16}$$

But from

$$H[K[u], u] \equiv 0$$

one finds

$$\int \bar{u}(\xi) d_\xi \eta[\xi, K, u] + \frac{\partial H}{\partial \tau} \int_0^1 \bar{u}(\xi) d_\xi \kappa[\xi, u] = 0, \tag{17}$$

where η is the functional associated with the differential of H when the τ is kept fixed. Furthermore, since $H[\tau, u] \equiv G[F[\xi, \tau, u]]$, it follows by Theorem 2 that H is a solution of equation (1′); i.e.,

$$\frac{\partial H}{\partial \tau}[\tau, u] + \int_0^1 f[\xi, u] d_\xi \eta[\xi, K, u] = 0.$$

Hence, substituting for $\bar{u}(\xi)$ in (17) $f[\xi, u]$, and making use of the preceding relation, one finds that (17) reduces to

$$\int_0^1 f[\xi, u] d_\xi \kappa[\xi, u] = 1$$

at every element for which $\partial H / \partial \tau |_{\tau_0, u_0} \neq 0$. Substituting $f[\xi, u]$ for $\bar{u}(\xi)$ in (16) and making use of the last relation and of the fact that F is a solution of equation (1′), one obtains finally that

$$\int_0^1 f[\xi, u] d_\xi \sigma[\xi, u] = \frac{\partial F}{\partial \tau} - \frac{\partial F}{\partial \tau} = 0$$

as desired. The following theorem has therefore been proved:

THEOREM 4. *If $F[\xi, \tau, u]$ is a solution of equation (1′) of the kind described in Theorem 1 and if G is an arbitrary functional of F*

$$G[F[\xi, \tau, u]] \equiv H[\tau, u].$$

such that H satisfies conditions (1), (2), (3) *and* (4), *then*

$$S[\xi, u] \equiv F[\xi, K[u], u]$$

is a solution of equation (2′) *where* $K[u]$ *is the unique solution of*

$$H[\tau, u] = 0$$

for τ.

Remark.—The arbitrary functional $H[\tau, u]$ should be chosen so that the implicit functional equation $H[\tau, u] = 0$ is easy to solve.

THEOREM 5. *If* $L[u]$ *is any solution of equation* (2′) *possessing a difference function* $\Lambda[u, \bar{u}; \tilde{u}]$, *then L is a continuous functional of* $S[\xi, u]$ *described in Theorem* 4 *where u is thought of as fixed.*

In the first place it is readily shown that if $L[u]$ is a solution of equation (2′), then $L[F[\xi, \tau, u]]$ is independent of τ. For, by Lemma 3

$$\frac{dL}{d\tau} = \int_0^1 \frac{\partial F}{\partial \tau}[\xi, \tau, u] d_\xi \mu[\xi, u],$$

where μ is the functional associated with the differential of $L[F[\xi, \tau, u]] \equiv M[\tau, u]$ with respect to the second argument. But

$$\frac{\partial F}{\partial \tau}[\xi, \tau, u] = -f[\xi, \tau, u],$$

so that

$$\frac{dL}{d\tau} = -\int_0^1 f[\xi, \tau, u] d_\xi \mu[\xi, \tau, u] = 0,$$

since by hypothesis L is a solution of equation (2′).

Since $L[F[\xi, \tau, u]]$ is independent of τ, it follows that

$$L[F[\xi, \tau, u]] = L[u] \qquad \text{for} \qquad \tau = \tau_0,$$

and

$$L[F[\xi, \tau, u]] = L[S[\xi, u]] \qquad \text{for} \qquad \tau = K[u].$$

Hence it follows for all functions u of the type considered that

$$L[u] = L[S[\xi, u]].$$

This completes the proof of the theorem.

As an illustration of the results of this section, consider the equation associated with (8),

$$\int_0^1 \left\{ \int_0^1 K[\eta, \zeta) u(\zeta) d\zeta \right\} d_\eta \varphi[\eta, u] = 0. \tag{18}$$

It has already been proved that a solution of equation (8) is given by

$$F[\xi, \tau, u] \equiv u(\xi) + \sum_{i=1}^{\infty} \varphi_i(\xi)(e^{-(\tau-\tau_0)/\lambda_i} - 1) \int_0^1 \varphi_i(\zeta) u(\zeta) d\zeta.$$

Take for the arbitrary function $G[F]$, the expression

$$\int_0^1 \varphi_k(\xi)F[\xi, \tau, u]d\xi, \quad \text{where} \quad \int_0^1 \varphi_k(\xi)u(\xi)d\xi \neq 0^*$$

and the function $H[\tau, u]$ is easily found to be

$$H[\tau, u] \equiv \int_0^1 \varphi_k(\xi)u(\xi)d\xi + (e^{-(\tau-\tau_0)/\lambda_k} - 1)\int_0^1 \varphi_k(\xi)u(\xi)d\xi$$
$$- \int_0^1 \varphi_k(\xi)u_0(\xi)d\xi = e^{-(\tau-\tau_0)/\lambda_k}\int_0^1 \varphi_k(\xi)u(\xi)d\xi - \int_0^1 \varphi_k(\xi)u_0(\xi)d\xi$$

so that

$$S[\xi, u] \equiv u(\xi) + \sum_{i=1}^{\infty}\varphi_i(\xi)\int_0^1 \varphi_i(\zeta)u(\zeta)d\zeta\left[\left(\frac{\int_0^1 \varphi_k(\xi)u_0(\xi)}{\int_0^1 \varphi_k(\xi)u(\xi)}\right)^{\lambda_k/\lambda_i} - 1\right]^\dagger$$

is the "general solution" of equation (18).

As some particular solutions, one may take

$$\int_0^1 \varphi_j(\xi)S[\xi, u]d\xi = \left(\frac{\int_0^1 \varphi_k(\xi)u_0(\xi)d\xi}{\int_0^1 \varphi_k(\xi)u(\xi)d\xi}\right)^{\lambda_k/\lambda_j}\int_0^1 \varphi_j(\xi)u(\xi)d\xi, \qquad j \neq k.$$

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* It is well known in the theory of integral equations that this expression could not be zero for all the characteristic functions unless u were identically zero.

† Cf. p. 203 of article by writer entitled "Functionals Invariant under One-parameter Continuous Groups of Transformations in the Space of Continuous Functions," *Proceedings of the National Academy of Sciences*, Vol. 6, No. 4, pp. 200–204, April, 1920.

ON THE ORDERING OF THE TERMS OF POLARS AND TRANSVECTANTS OF BINARY FORMS.

By L. Isserlis.

The reduction of transvectants depends on the possibility of inserting between any two terms of a transvectant a series of others such that any two consecutive terms possess the property which is technically known as adjacence. It is asserted without proof, that this is possible, by Gordan ("Vorlesungen über Invariantentheorie," Zweiter Band, § 42, S. 44), by Clebsch (Binäre Formen, § 53, S. 185), and by Grace and Young ("Algebra of Invariants," § 50, p. 51).

In 1908 I gave a short sketch of a method of effecting the actual ordering of the terms, and in the present paper I develop one such method in full and illustrate it by ordering the 56 terms of the transvectant

$$T = \{(ax)^m(bx)^n(cx)^p,\ (dx)^s(ex)^t\}^3.$$

I gladly acknowledge my obligation to Professor M. J. M. Hill, F.R.S., of the University of London. He drew my attention to the importance of the problem in his lectures on the Algebra of Invariants and indeed himself devised a method of ordering the terms of a polar differing somewhat from that given in Section 2 below.

§ 1. Definitions and Notation.

Two terms of a polar are said to be adjacent when they differ only in that one has a factor of the form $(\alpha_h x)(\alpha_k y)$ while in the other this factor is replaced by $(\alpha_h y)(\alpha_k x)$.

Two terms of a transvectant are said to be adjacent when they differ merely in the arrangement of the letters in a pair of symbolical factors. Two terms can be adjacent in two ways:

(i) $P\cdot(\alpha_i\beta_j)(\alpha_h\beta_k)$ and $P\cdot(\alpha_i\beta_k)(\alpha_h\beta_j)$,

(ii) $P\cdot(\alpha_i\beta_j)(\alpha_h x)$ and $P\cdot(\alpha_h\beta_j)(\alpha_i x)$,

where the letters α_1, α_2, $\cdots$ belong to one of the two forms of the transvectant and the letters β_1, β_2, $\cdots$ belong to the other form, while P represents a symbolic product.

We have to discuss a method of arranging the terms of (i) the ordinary

polar

$$P = \{(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}\}_{y^r}$$
$$= \left(y\frac{\partial}{\partial x}\right)^r\{(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}\},$$

if we omit a numerical factor, where $y(\partial/\partial x) \equiv y_1(\partial/\partial x_1) + y_2(\partial/\partial x_2)$.

(ii) the mixed polar

$$Q = \left(y_1\frac{\partial}{\partial x}\right)^{r_1}\left(y_2\frac{\partial}{\partial x}\right)^{r_2}\cdots\left(y_m\frac{\partial}{\partial x}\right)^{r_m}\{(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}\}$$

and (iii) the transvectant

$$T = [(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p},\ (b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}]^r$$

in such a manner that each term shall be adjacent to its neighbours.

It is to be noted that the rth polar of

$$f = (a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}$$

is

$$\frac{(n_1+n_2+\cdots+n_p-r)!}{(n_1+n_2+\cdots+n_p)!}\cdot P.$$

and that

$$\frac{(n_1+n_2+\cdots+n_p-r_1-r_2-\cdots-r_m)!}{(n_1+n_2+\cdots+n_p)!}Q$$

is a mixed polar of f, but in what follows these numerical factors and the numerical coefficients of the various terms will be omitted, because their values do not affect the ordering of the terms.

§ 2. To Order the Terms of the Polar

$$P = \left(y\frac{\partial}{\partial x}\right)^r[(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}]$$

Let D_s denote an operator which polarizes powers of (a_sx) only, with regard to y, then

$$P = (D_1+D_2+\cdots+D_p)^r[(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}].$$

Consider the terms which arise from operators which are alike except that one factor D_s of the first is replaced in the other by D_t, thus

$$D_1^{r_1}D_2^{r_2}\cdots D_s^{\lambda}D_t^{\mu}\cdots D_p^{r_p} \qquad \text{(I)}$$

and

$$D_1^{r_1}D_2^{r_2}\cdots D_s^{\lambda-1}D_t^{\mu+1}\cdots D_p^{r_p} \qquad \text{(II)}$$

where $r_1 + r_2 + \cdots + \lambda + \mu + \cdots + r_p = r$. The terms of the polar arising from these are

$$T_1 = (a_1x)^{n_1-r_1}(a_1y)^{r_1} \cdots (a_sx)^{n_s-\lambda}(a_sy)^{\lambda}(a_tx)^{n_t-\mu}(a_ty)^{\mu} \cdots (a_px)^{n_p-r_p}(a_py)^{r_p},$$

$$T_2 = (a_1x)^{n_1-r_1}(a_1y)^{r_1} \cdots (a_sx)^{n_s-\lambda+1}(a_sy)^{\lambda-1}(a_tx)^{n_t-\mu-1}(a_ty)^{\mu+1} \cdots (a_px)^{n_p-r_p}(a_py)^{r_p},$$

and are adjacent since the factors $(a_sy)(a_tx)$ of the first are replaced by $(a_ty)(a_sx)$ in the second. Call the operators I and II consecutive operators; we must therefore arrange the terms of

$$(D_1 + D_2 + \cdots + D_p)^r$$

consecutively. Now it is obvious that the ordered development of $(D_1 + D_2)^r$ is D_1^r, $D_1^{r-1}D_2$, $D_1^{r-2}D_2^2$, $\cdots$, D_2^r. We shall denote this by $(D_1 + D_2)^r$ and then $(D_2 + D_1)^r$ will stand for D_2^r, $D_2^{r-1}D_1$, $\cdots$, $D_2D_1^{r-1}$, D_1^r.

To obtain the ordered development of $(D_1 + D_2 + D_3)^r$ we first order $(D_1 + D_2)^r$. In the first term in which D_2 occurs, replace it by $D_2 + D_3$, in the second term in which it occurs replace it by $D_3 + D_2$, in the third by $D_2 + D_3$, and so on alternately. In this manner we obtain

$$D_1^r,\ D_1^{r-1}(D_2 + D_3),\ D_1^{r-2}(D_3 + D_2)^2,\ \cdots D_1^{r-2s}(D_3 + D_2)^{2s},\ D_1^{r-2s-1}(D_2 + D_3)^{2s+1},\ \cdots,$$

which becomes an ordered expansion of $(D_1 + D_2 + D_3)^r$ when each expression as $D_1^{r-2s}(D_3 + D_2)^{2s}$ has been expanded into a group of ordered terms of which the *last* is $D_1^{r-2s}D_2^{2s}$ and $D_1^{r-2s-1}(D_2 + D_3)^{2s+1}$ is expanded into an ordered group whose *first* term is $D_1^{r-2s-1}D_2^{2s+1}$ and this is consecutive to $D_1^{r-2s}D_2^{2s}$.

We shall prove by induction that this method is perfectly general. Let us assume that

$$(D_1 + D_2 + \cdots + D_p)^r$$

has been ordered. To order the terms of

$$(D_1 + D_2 + \cdots + D_p + D_{p+1})^r$$

replace D_p by $D_p + D_{p+1}$ in the first term of $(D_1 + D_2 + \cdots + D_p)^r$ in which it occurs, in the second term in which it occurs replace D_p by $D_{p+1} + D_p$, in the third by $D_p + D_{p+1}$ and so on alternately and we shall obtain the ordered expansion of

$$(D_1 + D_2 + \cdots + D_p + D_{p+1})^r.$$

For let

$$D_1^{\lambda_1}D_2^{\lambda_2} \cdots D_s^{\lambda_s}D_t^{\lambda_t} \cdots D_p^{\lambda_p} = T_{l-1},$$

$$D_1^{\lambda_1}D_2^{\lambda_2} \cdots D_s^{\lambda_s-1}D_t^{\lambda_t+1} \cdots D_p^{\lambda_p} = T_l$$

be two consecutive terms in the ordered expansion of $(D_1 + D_2 + \cdots + D_p)^r$ where $\sum\lambda = r$. When the substitutions of $D_p + D_{p+1}$ and $D_{p+1} + D_p$ for D_p have been made, then T_{l-1} and T_l become

$$T'_{l-1} = D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s} D_t^{\lambda_t} \cdots (D_p + D_{p+1})^{\lambda_p},$$
$$T'_l = D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s - 1} D_t^{\lambda_t + 1} \cdots (D_{p+1} + D_p)^{\lambda_p}.$$

On expanding the last factors we get two consecutively arranged groups of terms of the operator $(D_1 + D_2 + \cdots + D_p + D_{p+1})^r$. The last term of the first group is

$$D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s} D_t^{\lambda_t} \cdots D_{p+1}^{\lambda_p}.$$

The first term of the second group is

$$D_1^{\lambda_1} D_2^{\lambda_2} \cdots D_s^{\lambda_s - 1} D_t^{\lambda_t + 1} \cdots D_{p+1}^{\lambda_p}$$

and are themselves consecutive, so that the two groups together are a part of the ordered operator $(D_1 + D_2 + \cdots + D_{p+1})^r$. The above supposes that the suffixes s and t are neither of them equal to p, we must therefore consider specially the case in which T_{l-1} contains $D_s^{\lambda_s} D_p^{\lambda_p}$ and T_l contains $D_s^{\lambda_s - 1} D_p^{\lambda_p \pm 1}$.

Here

$$T_{l-1} = D_1^{\lambda_1} \cdots D_s^{\lambda_s} \cdots D_{p-1}^{\lambda_{p-1}} D_p^{\lambda_p},$$
$$T_l = D_1^{\lambda_1} \cdots D_s^{\lambda_s - 1} \cdots D_{p-1}^{\lambda_{p-1}} D_p^{\lambda_p \pm 1}.$$

After substitution

$$T'_{l-1} = D_1^{\lambda_1} \cdots D_s^{\lambda_s} \cdots D_{p-1}^{\lambda_{p-1}} (D_p + D_{p+1})^{\lambda_p},$$
$$T'_l = D_1^{\lambda_1} \cdots D_s^{\lambda_s - 1} \cdots D_{p-1}^{\lambda_{p-1}} (D_{p+1} + D_p)^{\lambda_p \pm 1}.$$

The last term of T'_{l-1} and the first term of T'_l are

$$D_1^{\lambda_1} \cdots D_s^{\lambda_s} \cdots D_{p-1}^{\lambda_{p-1}} D_{p+1}^{\lambda_p}$$

and

$$D_1^{\lambda_1} \cdots D_s^{\lambda_s - 1} \cdots D_{p-1}^{\lambda_{p-1}} D_{p+1}^{\lambda_p \pm 1},$$

and so these are consecutive.

Now $(D_1 + D_2)^r$ and $(D_1 + D_2 + D_3)^r$ have been ordered, hence $(D_1 + D_2 + D_3 + D_4)^r$ can be ordered, and so on. It is important in what follows to notice that when expanded in this way $(D_1 + D_2 + \cdots + D_p)^r$ commences with D_1^r and ends with D_k^r where k is one of the integers 2, 3, 4, $\cdots$ p and can be made any one of them we please.*

* The method adopted by Professor Hill for ordering $(D_1 + D_2 + \cdots + D_p)^r$ is as follows. The ordered arrangement of $(D_1 + D_2)^r$ is $D_1^r, D_1^{r-1} D_2, \cdots, D_2^r$, call this the direct order for $(D_1 + D_2)^r$ and $D_2^r, D_2^{r-1} D_1, \cdots, D_1^r$ the reverse order. A direct order for

§ 3. On Ordering the Terms of a Mixed Polar.

Let the mixed polar be

$$\left(y\frac{\partial}{\partial x}\right)^{r_k}_{y=y_k}\left(y\frac{\partial}{\partial x}\right)^{r_{k-1}}_{y=y_{k-1}}\cdots\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}\{(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}\}$$

and let the result of ordering

$$\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}\{(b_1x)^{m_1}\cdots(b_qx)^{m_q}\},$$

as in § 2, be $A_1+A_2+\cdots+A_{2s-1}+A_{2s}+\cdots$.

We may put

$$\left(y\frac{\partial}{\partial x}\right)^{r_2}=(D_1+D_2+\cdots+D_q)^{r_2}$$
$$=D_1^{r_2}+\cdots+D_t^{r_2}$$

when ordered as in § 2 where t is one of the numbers $2, 3, \cdots q$. Then the ordered development of

$$\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}\{(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}\}$$

is

$$(D_1^{r_2}+\cdots+D_t^{r_2})_{y=y_2}A_1+(D_t^{r_2}+\cdots+D_1^{r_2})_{y=y_2}A_2+\cdots$$
$$+(D_1^{r_2}+\cdots+D_t^{r_2})_{y=y_2}A_{2s-1}+(D_t^{r_2}+\cdots+D_1^{r_2})_{y=y_2}A_{2s}+\cdots.$$

where $D_t^{r_2}+\cdots+D_1^{r_2}$ is $D_1^{r_2}+\cdots+D_t^{r_2}$ written in reverse order. To prove this we first observe that by § 2

$$(D_1^{r_2}+\cdots+D_t^{r_2})_{y=y_2}A_{2s-1}$$

is a group of consecutive terms; it will therefore be sufficient to prove that $D_t^{r_2}A_{2s-1}$ and $D_t^{r_2}A_{2s}$ are consecutive terms.

$(D_1+D_2+D_3)^r$ is

$(D_1+D_2)^r$ in direct order $+$ $[(D_1+D_2)^{r-1}$ in reverse order$]D_3$

$+$ $[(D_1+D_2)^{r-2}$ in direct order$]D_3^2+\cdots$.

When the direct order for $(D_1+D_2+D_3)^r$ is established, we get a direct order for $(D_1+D_2+D_3+D_4)^r$ by writing $(D_1+D_2+D_3)^r$ in direct order

$+$ $[(D_1+D_2+D_3)^{r-1}$ in reverse order$]D_4$
$+$ $[(D_1+D_2+D_3)^{r-2}$ in direct order$]D_4^2$

and so on.

It can be proved by induction that this method is general. From this can be deduced the ordering of a transvectant in which one of the two forms contains only a single binary form.

Now A_{2s-1} and A_{2s} are either of the form

$$A_{2s-1} = H(b_t x)^{m_t-\lambda_t}(b_t y_1)^{\lambda_t}(b_u x)(b_v y_1),$$
$$A_{2s} \;\;\, = H(b_t x)^{m_t-\lambda_t}(b_t y_1)^{\lambda_t}(b_v x)(b_u y_1),$$

or of the form indicated in the special case below, H being a symbolic product not involving $(b_t x)$.

Hence,

$$D_t^{r_2} A_{2s-1} = H(b_t x)^{m_t-\lambda_t-r_2}(b_t y_1)^{\lambda_t}(b_t y_2)^{r_2}(b_u x)(b_v y_1)$$

and

$$D_t^{r_2} A_{2s} = H(b_t x)^{m_t-\lambda_t-r_2}(b_t y_1)^{\lambda_t}(b_t y_2)^{r_2}(b_v x)(b_u y_1)$$

and are consecutive.

Special Case.—We must verify that this still holds in the special case in which t is u or v. We may, when $t = u$, write

$$A_{2s-1} = H(b_u x)^{m_u-\lambda_u}(b_u y_1)^{\lambda_u}(b_v y_1),$$
$$A_{2s} \;\;\, = H(b_u x)^{m_u-\lambda_u-1}(b_u y_1)^{\lambda_u+1}(b_v x),$$

so that

$$D_u^{r_2} A_{2s-1} = H(b_u x)^{m_u-\lambda_u-r_2}(b_u y_1)^{\lambda_u}(b_u y_2)^{r_2}(b_v y_1) = K(b_u x)(b_v y_1)$$

and

$$D_u^{r_2} A_{2s} = H(b_u x)^{m_u-\lambda_u-r_2-1}(b_u y_1)^{\lambda_u+1}(b_u y_2)^{r_2}(b_v x) = K(b_u y_1)(b_v x),$$

so that the adjacence holds in this case also.

Thus the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}(b_1 x)^{m_1}(b_2 x)^{m_2}\cdots(b_q x)^{m_q}$$

can be ordered; denote the result by

$$B_1 + B_2 + \cdots + B_{2s-1} + B_{2s} + \cdots.$$

Let $D_1^{r_3} + \cdots + D_k^{r_3}$ be the ordered development of the operator in $\left(y\frac{\partial}{\partial x}\right)^{r_3}_{y=y_3}$ or of $(D_1 + D_2 + \cdots + D_q)^{r_3}$ and let $D_k^{r_3} + \cdots + D_1^{r_3}$ be the same reversed.

Then the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_3}_{y=y_3}\left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}(b_1 x)^{m_1}(b_2 x)^{m_2}\cdots(b_q x)^{m_q}$$

will be ordered if we expand it in the form

$$(D_1^{r_3} + \cdots + D_k^{r_3})_{y=y_3}B_1 + (D_k^{r_3} + \cdots + D_1^{r_3})_{y=y_3}B_2 + \cdots$$
$$+ (D_1^{r_3} + \cdots + D_k^{r_3})_{y=y_3}B_{2s-1} + (D_k^{r_3} + \cdots + D_1^{r_3})_{y=y_3}B_{2s} + \cdots.$$

As before, $(D_1^{r_3} + \cdots + D_k^{r_3})_{y=y_3}B_{2s-1}$ is a group of consecutive terms;

it will therefore be sufficient to prove that

$$D_k^{r_3}B_{2s-1} \quad \text{and} \quad D_k^{r_3}B_{2s}$$

are consecutive terms.

Now B_{2s-1} and B_{2s} are either of the form

$$\left.\begin{aligned} B_{2s-1} &= H(b_kx)^{m_k-\lambda_k-\mu_k}(b_ky_1)^{\lambda_k}(b_ky_2)^{\mu_k}(b_ux)(b_vy) \\ B_{2s} &= H(b_kx)^{m_k-\lambda_k-\mu_k}(b_ky_1)^{\lambda_k}(b_ky_2)^{\mu_k}(b_vx)(b_uy) \end{aligned}\right\} \text{ where } y \text{ is either } y_1 \text{ or } y_2$$

or of the form considered in the special case below. Therefore,

$$\begin{aligned} D_k^{r_3}B_{2s-1} &= H(b_kx)^{m_k-\lambda_k-\mu_k-r_3}(b_ky_1)^{\lambda_k}(b_ky_2)^{\mu_k}(b_ky_3)^{r_3}(b_ux)(b_vy), \\ D_k^{r_3}B_{2s} &= H(b_kx)^{m_k-\lambda_k-\mu_k-r_3}(b_ky_1)^{\lambda_k}(b_ky_2)^{\mu_k}(b_ky_3)^{r_3}(b_vx)(b_uy) \end{aligned}$$

and are consecutive.

Special Case.—The special case in which k is u or v can be treated as before; thus suppose k is u and that in (b_uy), (b_vy) y stands for y_2, then

$$\begin{aligned} B_{2s-1} &= H(b_ux)^{m_u-\lambda_u-\mu_u}(b_uy_1)^{\lambda_u}(b_uy_2)^{\mu_u}(b_vy_2), \\ B_{2s} &= H(b_ux)^{m_u-\lambda_u-\mu_u-1}(b_uy_1)^{\lambda_u}(b_uy_2)^{\mu_u+1}(b_vx), \end{aligned}$$

so that

$$D_k^{r_3}B_{2s-1} = H(b_ux)^{m_u-\lambda_u-\mu_u-r_3}(b_uy_1)^{\lambda_u}(b_uy_2)^{\mu_u}(b_uy_3)^{r_3}(b_vy_2) = K(b_ux)(b_vy_2)$$

and

$$D_k^{r_3}B_{2s} = H(b_ux)^{m_u-\lambda_u-\mu_u-r_3-1}(b_uy_1)^{\lambda_u}(b_uy_2)^{\mu_u+1}(b_uy_3)^{r_3}(b_vx) = K(b_uy_2)(b_vx),$$

so that the adjacence holds in this case.

We shall show that this method of ordering the mixed polar is perfectly general.

General Case.—Let the ordered development of the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_{k-1}}_{y=y_{k-1}} \cdots \left(y\frac{\partial}{\partial x}\right)^{r_2}_{y=y_2}\left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}$$

be

$$P_1 + P_2 + \cdots + P_{2s-1} + P_{2s} + \cdots$$

and let

$$\left(y\frac{\partial}{\partial x}\right)^{r_k}_{y=y_k} \text{ or } (D_1 + D_2 + \cdots + D_q)^{r_k} \equiv D_1^{r_k} + \cdots + D_t^{r_k}$$

when ordered, $D_t^{r_k} + \cdots + D_1^{r_k}$ being the same written in reverse order. Then the ordered arrangement of the mixed polar

$$\left(y\frac{\partial}{\partial x}\right)^{r_k}_{y=y_k}\left(y\frac{\partial}{\partial x}\right)^{r_{k-1}}_{y=y_{k-1}} \cdots \left(y\frac{\partial}{\partial x}\right)^{r_1}_{y=y_1}(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}$$

is

$$(D_1^{r_k} + \cdots + D_t^{r_k})P_1 + (D_t^{r_k} + \cdots + D_1^{r_k})P_2 + \cdots$$
$$+ (D_1^{r_k} + \cdots + D_t^{r_k})P_{2s-1} + (D_t^{r_k} + \cdots + D_1^{r_k})P_{2s} + \cdots.$$

As before, we must prove that

$$D_t^{r_k}P_{2s-1} \quad \text{and} \quad D_t^{r_k}P_{2s}$$

are consecutive. Let

$$P_{2s-1} = H(b_tx)^{m_t-\lambda_1-\lambda_2-\cdots-\lambda_{k-1}}(b_ty_1)^{\lambda_1}(b_ty_2)^{\lambda_2}\cdots(b_ty_{k-1})^{\lambda_{k-1}}(b_ux)(b_vy_i),$$
$$P_{2s} = \text{———————————————}(b_uy_i)(b_vx).$$

Then

$$D_t^{r_k}P_{2s-1} = H(b_tx)^{m_t-\lambda_1-\lambda_2-\cdots-\lambda_{k-1}-r_k}(b_ty_1)^{\lambda_1}(b_ty_2)^{\lambda_2}\cdots(b_ty_{k-1})^{\lambda_{k-1}}(b_ty_k)^{r_k}$$
$$(b_ux)(b_vy_i),$$
$$D_t^{r_k}P_{2s} = \text{———————————————}(b_uy_i)(b_vx),$$

and are consecutive.

In the special case in which u is t

$$P_{2s-1} = H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}}(b_ty_1)^{\lambda_1}(b_ty_2)^{\lambda_2}\cdots(b_ty_i)^{\lambda_i}\cdots(b_ty_{k-1})^{\lambda_{k-1}}(b_vy_i),$$
$$P_{2s} = H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}-1}(b_ty_1)^{\lambda_1}\cdots(b_ty_i)^{\lambda_i+1}\cdots(b_ty_{k-1})^{\lambda_{k-1}}(b_vx),$$

so that

$$D_t^{r_k}P_{2s-1} = H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}-r_k}(b_ty_1)^{\lambda_1}\cdots(b_ty_i)^{\lambda_i}\cdots(b_ty_{k-1})^{\lambda_{k-1}}(b_ty_k)^{r_k}(b_vy_i)$$
$$= K(b_tx)(b_vy_i),$$
$$D_t^{r_k}P_{2s} = H(b_tx)^{m_t-\lambda_1-\cdots-\lambda_{k-1}-r_k-1}(b_ty_1)^{\lambda_1}\cdots(b_ty_i)^{\lambda_i+1}$$
$$\cdots(b_ty_{k-1})^{\lambda_{k-1}}(b_ty_k)^{r_k}(b_vx) = K(b_ty_i)(b_vx),$$

so that the adjacence holds in this case. Hence the terms of a mixed polar can be ordered by the above method in the general case.

§ 4. THE ORDERING OF THE TERMS OF A TRANSVECTANT.

When it is required to calculate the transvectant

$$T = \{(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p},\quad (b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}\}^r$$

we first order the terms of the polar

$$\{(a_1x)^{n_1}(a_2x)^{n_2}\cdots(a_px)^{n_p}\}_{y^r}.$$

In the result replace y_1 by h_2 and y_2 by $-h_1$ where

$$(hx)^{m_1+m_2+\cdots+m_q} \equiv (b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}$$

and multiply the new result by $(hx)^{m_1+m_2+\cdots+m_p-r}$. This gives T when all the factors containing h have been evaluated. A factor such as $(a_1h)^{\lambda_1}\cdots(a_ph)^{\lambda_p}(hx)^{m_1+\cdots+m_q-r}$ is evaluated by polarizing $(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}$, λ_1 times with respect to y_1, λ_2 times with respect to y_2, $\cdots$, λ_p times with respect to y_p and then putting

$$\left.\begin{matrix} y_{11} = -a_{12} \\ y_{12} = a_{11} \end{matrix}\right\} \left.\begin{matrix} y_{21} = -a_{22} \\ y_{22} = a_{21} \end{matrix}\right\} \cdots \begin{matrix} y_{p1} = -a_{p_2} \\ y_{p2} = a_{p_1} \end{matrix}$$

or symbolically

$$T = \sum (a_1x)^{n_1-\lambda_1}\cdots(a_px)^{n_p-\lambda_p}\left(y_1\frac{\partial}{\partial x}\right)^{\lambda_1}\cdots\left(y_p\frac{\partial}{\partial x}\right)^{\lambda_p}\{(b_1x)^{m_1}(b_2x)^{m_2}\cdots(b_qx)^{m_q}\}$$

$$\lambda_1+\lambda_2+\cdots+\lambda_q = r.$$

Let D_{ks} be an operator which polarizes (b_sx) only, with respect to y_k, then we may put

$$\left(y_1\frac{\partial}{\partial x}\right)^{\lambda_1} = (D_{11}+D_{12}+\cdots+D_{1q})^{\lambda_1},$$

$$\left(y_2\frac{\partial}{\partial x}\right)^{\lambda_2} = (D_{21}+D_{22}+\cdots+D_{2q})^{\lambda_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(y_p\frac{\partial}{\partial x}\right)^{\lambda_p} = (D_{p1}+D_{p2}+\cdots+D_{pq})^{\lambda_p}.$$

T is thus arranged as a sum of groups of terms. The factors preceding the operators are already arranged so that if one is $H(a_sx)^u(a_tx)^v$ the next is $H(a_sx)^{u-1}(a_tx)^{v+1}$ reducible to $K(a_sx)$ and $K(a_tx)$ where H and K are products of linear factors only.

We must now develop the operators of form

$$\left(y\frac{\partial}{\partial x}\right)^{\lambda_1}\cdots y\frac{\partial}{\partial x}\biggr)^{\lambda_p}$$

or

$$(D_{11}+D_{12}+\cdots+D_{1q})^{\lambda_1}\cdots(D_{p1}+\cdots+D_{pq})^{\lambda_q}$$

so that when the operations have been carried out

(i) in each group every term is adjacent to its neighbors,

(ii) the last term of any group is adjacent to the first term of the next group.

§ 5. Let the transvectant

$$\begin{aligned} T = \cdots &+ H(a_s x)(D_{11} + \cdots + D_{1q})^{\lambda_1} \\ &\qquad \cdots (D_{s1} + \cdots + D_{sq})^{\lambda_s}(D_{t1} + \cdots + D_{tq})^{\lambda_t} \\ &\qquad \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} \\ &+ H(a_t x)(D_{11} + \cdots + D_{1q})^{\lambda_1} \\ &\qquad \cdots (D_{s1} + \cdots + D_{sq})^{\lambda_s+1}(D_{t1} + \cdots + D_{tq})^{\lambda_t-1} \\ &\qquad \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} \\ &+ \cdots, \end{aligned}$$

where H is a product of factors of type (ax). Denote the expanded forms of the operators in the terms written down above by $P_1 + P_2 + \cdots + P_\mu$ and $Q_1 + Q_2 + \cdots + Q_\nu$. Then

(i) $\left.\begin{matrix} H(a_sx)\cdot P_\mu \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \\ H(a_tx)\cdot Q_1 \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \end{matrix}\right\}$ must be adjacent terms of T

and

(ii) $\left.\begin{matrix} H(a_sx)\cdot P_k \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \\ H(a_sx)\cdot P_{k+1} \cdot (b_1x)^{m_1} \cdots (b_qx)^{m_q} \end{matrix}\right\}$ must be adjacent terms of T.

One way of satisfying (i) is to expand the operators so that the first operator $(D_{11} + \cdots + D_{1q})^{\lambda_1} \cdots (D_{p1} + \cdots + D_{pq})^{\lambda_p}$ always starts with $D_{11}^{\lambda_1}D_{21}^{\lambda_2} \cdots D_{p1}^{\lambda_p}$ (i.e., a product of D's with second suffix 1) and always ends with $D_{1q}^{\lambda_1} \cdots D_{sq}^{\lambda_s}D_{tq}^{\lambda_t} \cdots D_{pq}^{\lambda_p}$ (i.e., a product of D's with second suffix q). For then we may write the second operator as

$$(D_{1q} + \cdots + D_{11})^{\lambda_1} \cdots (D_{sq} + \cdots + D_{s1})^{\lambda_s+1}(D_{tq} + \cdots + D_{t1})^{\lambda_t-1} \cdots (D_{pq} + \cdots + D_{p1})^{\lambda_p}$$

so that the last term in the first group is

$$H(a_sx)D_{1q}^{\lambda_1} \cdots D_{sq}^{\lambda_s}D_{tq}^{\lambda_t} \cdots D_{pq}^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\}_{y_1=a_1, y_2=a_2, \cdots}$$

and the first operator in the second group is

$$H(a_tx)D_{1q}^{\lambda_1} \cdots D_{sq}^{\lambda_s+1}D_{tq}^{\lambda_t-1} \cdots D_{pq}^{\lambda_p}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\}_{y_1=a_1, y_2=a_2, \cdots}$$

and these are adjacent for they are of the form $K(a_sx)(b_qa_t)$ and $K(a_tx)(b_qa_s)$ where K is the same in both.

Condition (ii) can be satisfied in either of the following two ways. (*a*) If $P_k = \Delta Duv$ and $P_{k+1} = \Delta Duw$ where Δ is the same product of D's in both, or (*b*) if $P_k = \Delta DuvDtw$ and $P_{k+1} = \Delta DuwDtv$. For if (*a*) holds,

$$H(a_sx)P_k\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} = K(b_wx)(b_va_u)$$

and

$$H(a_sx)P_{k+1}\{(b_1x)^{m_1} \cdots (b_qx)^{m_q}\} = K(b_vx)(b_wa_u),$$

where K is the same in both. And if (b) holds,

$$H(a_sx)P_k\{(b_1x)^{m_1}\cdots(b_qx)^{m_q}\}=K(b_va_u)(b_wa_t)$$
$$H(a_sx)P_{k+1}\{(b_1x)^{m_1}\cdots(b_qx)^{m_q}\}=K(b_wa_u)(b_va_t),$$

where K is the same in both. Therefore in both cases we get adjacent terms of the transvectant.

§ 6. Thus we can order the terms of the transvectant if we can expand

$$(D_{11}+\cdots+D_{1q})^{\lambda_1}(D_{21}+\cdots+D_{2q})^{\lambda_2}\cdots(D_{p1}+\cdots+D_{pq})^{\lambda_p},$$

so that (i) it starts with $D_{11}^{\lambda_1}D_{21}^{\lambda_2}\cdots D_{p1}^{\lambda_p}$; (ii) it ends with $D_{1q}^{\lambda_1}D_{2q}^{\lambda_2}\cdots D_{pq}^{\lambda_p}$; and (iii) any two consecutive terms are either of the form ΔDuv, ΔDuw or of the form $\Delta Duv\ Dtw, \Delta Duw Dtv$. We shall show later* that the general case can be deduced from the case $q=2$. We shall therefore commence with this case. So we must arrange

$$(D_{11}+D_{12})^{\lambda_1}(D_{21}+D_{22})^{\lambda_2}\cdots(D_{p1}+D_{p2})^{\lambda_p},$$

so as to satisfy conditions (i), (ii), (iii) above. Omitting numerical coefficients we expand $(D_{11}+D_{12})^{\lambda_1}$ in the form $D_{11}^{\lambda_1}+D_{11}^{\lambda_1-1}D_{12}+D_{11}^{\lambda_1-2}D_{12}^2+\cdots+D_{12}^{\lambda_1}$. We shall find it convenient to write for this

$$a_1+a_2+\cdots+a_{\lambda_1+1}.$$

Similarly, take

$$(D_{21}+D_{22})^{\lambda_2}=D_{21}^{\lambda_2}+D_{21}^{\lambda_2-1}D_{22}+\cdots+D_{22}^{\lambda_2}$$
$$=b_1+b_2+\cdots+b_{\lambda_2+1},$$

and so on. We are now concerned with the development of $(a_1+a_2+\cdots+a_{\lambda_1+1})(b_1+b_2+\cdots+b_{\lambda_2+1})(c_1+c_2+\cdots+c_{\lambda_3+1})\cdots$ in such a manner that (i) it starts with $a_1b_1c_1\cdots$; (ii) it ends with $a_{\lambda_1+1}b_{\lambda_2+1}c_{\lambda_3+1}\cdots$; and (iii) two consecutive terms are either of the form Pa_k and Pa_{k+1} where P is the same in both, or of the form Pa_kb_l and $Pa_{k+1}b_{l-1}$. Of course any of the letters $a, b, c, \cdots$ may be used instead of a or b. If the first form is used the adjacence of the corresponding terms of the transvectant is of the same kind as that of $(\alpha\beta)(\gamma x)\cdot K$ and $(\alpha\gamma)(\beta x)\cdot K$, but the second form produces terms whose adjacence is of the same kind as that of $(\alpha\beta)(\gamma\delta)K$ and $(\alpha\delta)(\gamma\beta)K$ for

$$a_k=D_{11}^{\lambda_1-k+1}D_{12}^{k-1},$$
$$b_l=D_{12}^{\lambda_2-l+1}D_{22}^{l-1},$$

so that if $Pa_k=MD_{11}$, then $Pa_{k+1}=MD_{12}$. But if $Pa_kb_l=MD_{11}D_{22}$, then $Pa_{k+1}b_{l-1}=MD_{12}D_{21}$, showing that condition (iii) is satisfied.

* § 9.

First let all the exponents or all the exponents but one be even (in the latter case put the factor $(D_{s1}+D_{s2})^{\lambda_s}$ say, with the odd exponent λ_s, first). Multiply out by the rule

$$\begin{aligned}(u+v+\cdots+x+y)(p+q+r+\cdots)\\ &= (u+v+\cdots+x+y)p+(y+x+\cdots+v+u)q\\ &\qquad\qquad+(u+v+\cdots+x+y)r+\cdots\\ &= up+vp+\cdots+yp+yq+xq+\cdots+uq+ur+vr+\cdots,\end{aligned}$$

and if there are more than two factors, the first two are to be multiplied together by the rule and then the result is to be multiplied by the third factor in accordance with the rule and so on. Thus

$$\begin{aligned}(a_1+a_2+\cdots+a_{\lambda_1+1})&(b_1+b_2+\cdots+b_{\lambda_2+1})(c_1+c_2+\cdots+c_{\lambda_3+1})\\ &= [(a_1+a_2+\cdots+a_{\lambda_1+1})b_1+(a_{\lambda_1+1}+\cdots+a_1)b_2\\ &\qquad+\cdots+(a_1+\cdots+a_{\lambda_1+1})b_{\lambda_2+1}](c_1+\cdots+c_{\lambda_3+1})\end{aligned}$$

in which b_{λ_2+1} multiplies the a's in direct order since λ_2+1 is odd. The complete product is

$$\begin{aligned}(a_1b_1+\cdots+a_{\lambda_1+1}b_{\lambda_2+1})c_1+(a_{\lambda_1+1}b_{\lambda_2+1}+\cdots+a_1b_1)c_2+\cdots\\ +(a_1b_1+\cdots+a_{\lambda_1+1}b_{\lambda_2+1})c_{\lambda_3+1},\end{aligned}$$

ending thus since λ_3+1 is odd.

It is clear that with any number of factors this rule will give a product commencing with $a_1b_1c_1\cdots$ and ending with $a_{\lambda_1+1}b_{\lambda_2+1}c_{\lambda_3+1}\cdots$ and any two consecutive terms will be of the form Pa_k, Pa_{k+1}.

§ 7. The case in which several of the exponents are odd cannot be done so simply. It is sufficient to consider the expansion of

$$(a_1+\cdots+a_{\lambda_1+1})(b_1+\cdots+b_{\lambda_2+1})(c_1+c_2+\cdots+c_{\lambda_3+1})\cdots,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are *all* odd, for when this has been multiplied out so as to satisfy the conditions (i), (ii), (iii) of § 6, the product can be multiplied in turn by each of the factors containing an odd number of terms without any difficulty.

So we consider

$$(a_1+\cdots+a_{2l})(b_1+\cdots+b_{2m})(c_1+c_2+\cdots+c_{2n})\cdots,$$

where there is an even number of terms in each factor.

Now $(a_1+a_2)(b_1+b_2)$ may be expanded by the scheme

$$a_1\begin{cases}b_1\\ b_2\end{cases}$$
$$a_2\begin{cases}b_1\\ b_2\end{cases}$$

i.e., $a_1b_1+a_1b_2+a_2b_1+a_2b_2$.

This satisfies the conditions (i), (ii), (iii) and the two middle terms will ultimately produce terms of the transvectant whose adjacence is of the same kind as that of $K(\alpha\beta)(\gamma\delta)$ and $K(\alpha\delta)(\gamma\beta)$. For $(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)$ we use the scheme

$$a_1\begin{cases} b_1\begin{cases} c_1 \\ c_2 \end{cases} \\ b_2\begin{cases} c_2 \\ c_1 \end{cases} \end{cases}$$

$$a_2\begin{cases} b_1\begin{cases} c_1 \\ c_2 \end{cases} \\ b_2\begin{cases} c_1 \\ c_2 \end{cases} \end{cases}$$

or $a_1b_1c_1 + a_1b_1c_2 + a_1b_2c_2 + a_1b_2c_1 + a_2b_1c_1 + a_2b_1c_2 + a_2b_2c_1 + a_2b_2c_2$ where again all the conditions are satisfied and the adjacence of the middle terms is as in the previous case.

Similarly

$$(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)(d_1 + d_2)$$

is arranged by the scheme

$$a_1\begin{cases} b_1\begin{cases} c_1\begin{cases} d_1 \\ d_2 \end{cases} \\ c_2\begin{cases} d_2 \\ d_1 \end{cases} \end{cases} \\ b_2\begin{cases} c_2\begin{cases} d_1 \\ d_2 \end{cases} \\ c_1\begin{cases} d_2 \\ d_1 \end{cases} \end{cases} \end{cases}$$

$$a_2\begin{cases} b_1\begin{cases} c_1\begin{cases} d_1 \\ d_2 \end{cases} \\ c_2\begin{cases} d_2 \\ d_1 \end{cases} \end{cases} \\ b_2\begin{cases} c_1\begin{cases} d_1 \\ d_2 \end{cases} \\ c_2\begin{cases} d_1 \\ d_2 \end{cases} \end{cases} \end{cases}$$

and so on, where in any vertical column (say the fourth) the entries d_1, d_2 and d_2, d_1 occur alternately *except that the last two entries are alike.*

§ 8. Now consider the product

$$(a_1 + a_2 + a_3 + a_4)(b_1 + b_2 + b_3 + b_4) = AB.$$

Bracket the last two terms in each bracket together, thus:

$$AB = (a_1 + a_2 + \overline{a_3 + a_4})(b_1 + b_2 + \overline{b_3 + b_4}),$$

and so begin by treating the product as though the factors contained an odd number of terms. So

$$
\begin{aligned}
AB &= b_1(a_1 + a_2 + \overline{a_3 + a_4}) + b_2(\overline{a_4 + a_3} + a_2 + a_1) \\
&\qquad\qquad\qquad\qquad + (b_3 + b_4)(a_1 + a_2 + \overline{a_3 + a_4}) \\
&= b_1a_1 + b_1a_2 + b_1a_3 + b_1a_4 + b_2a_4 + b_2a_3 + b_2a_2 + b_2a_1 \\
&\qquad\qquad\qquad + a_1(b_3 + b_4) + a_2(b_4 + b_3) + (a_3 + a_4)(b_3 + b_4).
\end{aligned}
$$

Now we can deal with $(a_3 + a_4)(b_3 + b_4)$. It will commence with a_3b_3 which will be "consecutive" to a_2b_3, where we use the word consecutive for brevity to denote that these operator terms are correctly ordered for producing adjacent terms of the transvectant, and it will end with a_4b_4.

In the same way

$$
\begin{aligned}
&(a_1 + a_2 + \overline{a_3 + a_4})(b_1 + b_2 + \overline{b_3 + b_4})(c_1 + c_2 + \overline{c_3 + c_4}) \\
&\qquad = [b_1a_1 + 10 \text{ ordered terms} + a_2b_3 + a_3a_4(b_3 + b_4)]c_1 \\
&\qquad\quad + [(b_4 + b_3)(a_4 + a_3) + a_2b_3 + 10 \text{ ordered terms} + b_1a_1]c_2 \\
&\qquad\quad + a_1b_1(c_3 + c_4) + b_1a_2(c_4 + c_3) + \cdots + a_2b_3(c_4 + c_3) \\
&\qquad\quad + (c_3 + c_4)(b_3 + b_4)(a_3 + a_4)
\end{aligned}
$$

can be consecutively arranged, and starts with $a_1b_1c_1$ and ends with $a_4b_4c_4$.

This can be generalized.

$$
\begin{aligned}
&(a_1 + a_2 + \cdots + a_{2p-2} + \overline{a_{2p-1} + a_{2p}})(b_1 + b_2 + \cdots + b_{2q-2} + \overline{b_{2q-1} + b_{2q}}) \\
&\quad = b_1(a_1 + \cdots + a_{2p}) + b_2(a_{2p} + \cdots + a_1) + \cdots + b_{2q-2}(a_{2p} + \cdots + a_1) \\
&\qquad + (b_{2q-1} + b_{2q})(a_1 + a_2 + \cdots + a_{2p-2} + \overline{a_{2p-1} + a_{2p}}).
\end{aligned}
$$

The last product is

$$
\begin{aligned}
a_1(b_{2q-1} + b_{2q}) + a_2(b_{2q} + b_{2q-1}) + \cdots + a_{2p-2}(b_{2q} + b_{2q-1}) \\
+ (b_{2q-1} + b_{2q})(a_{2p-1} + a_{2p})
\end{aligned}
$$

and therefore ends with $a_{2p}b_{2q}$.

Similarly

$$
\begin{aligned}
&(a_1 + a_2 + \cdots + a_{2p})(b_1 + \cdots + b_{2q})(c_1 + c_2 + \cdots + c_{2r-2} + c_{2r-1} + c_{2r}) \\
&\quad = (a_1b_1 + \cdots + a_{2p-2}b_{2q-1} + \overline{b_{2q-1} + b_{2q}}\ \overline{a_{2p-1} + a_{2p}})(c_1 + \cdots + c_{2r}) \\
&\quad = c_1(a_1b_1 + \cdots + a_{2p-2}b_{2q-1} + \cdots + b_{2q}a_{2p}) \\
&\qquad + c_2 \text{ (same reversed)} \\
&\qquad + c_3 \text{ (same direct)} \\
&\qquad + \cdots \\
&\qquad + c_{2r-2} \text{ (same reversed)} \\
&\qquad + (c_{2r-1} + c_{2r})(a_1b_1 + \cdots + a_{2p-2}b_{2q-1} + \overline{b_{2q-1} + b_{2q}}\ \overline{a_{2p-1} + a_{2p}})
\end{aligned}
$$

and the last line is

$$a_1b_1(c_{2r-1}+c_{2r})+\cdots+a_{2p-2}b_{2q-1}(c_{2r}+c_{2r-1})\\ +(c_{2r-1}+c_{2r})(b_{2q-1}+b_{2q})(a_{2p-1}+a_{2p});$$

therefore when the product in the preceding line is ordered, the whole product now under consideration will be ordered and starts with $a_1b_1c_1$ and ends with $a_{2p}b_{2q}c_{2r}$.

§ 9. We can therefore in all cases "order" the operator

$$A=(D_{11}+D_{12})^{\lambda_1}(D_{21}+D_{22})^{\lambda_2}\cdots(D_{p1}+D_{p2})^{\lambda_p}.$$

The transition from this to the operator

$$B=(D_{11}+D_{12}+D_{13})^{\lambda_1}(D_{21}+D_{22}+D_{23})^{\lambda_2}\cdots(D_{p1}+D_{p2}+D_{p3})^{\lambda_p}$$

is effected as follows:

In the *last* term of A each term of the form D_{k2} is replaced by $D_{k2}+D_{k3}$,
In the term before by $D_{k3}+D_{k2}$,
In the term before that by $D_{k2}+D_{k3}$,

and so on where $k=1, 2, \cdots p$.

Then since A begins with $D_{11}^{\lambda_1}\cdots D_{p1}^{\lambda_p}$ and ends with $D_{12}^{\lambda_1}\cdots D_{p2}^{\lambda_p}$, therefore B begins with $D_{11}^{\lambda_1}\cdots D_{p1}^{\lambda_p}$ and ends with $(D_{12}+D_{13})^{\lambda_1}\cdots(D_{p2}+D_{p3})^{\lambda_p}$, i.e., B ends with $D_{13}^{\lambda_1}\cdots D_{p3}^{\lambda_p}$.

Let two consecutive terms in A be

$$\Delta D_{12}^{\mu_1}D_{22}^{\mu_2}\cdots D_{p2}^{\mu_p} \quad\text{and}\quad \Delta' D_{12}^{\mu_1'}D_{22}^{\mu_2'}\cdots D_{p2}^{\mu_p'} \text{ where } \Delta,\ \Delta'$$

are products of D's whose second suffix is 1. These become in B say,

$$\Delta(D_{12}+D_{13})^{\mu_1}(D_{22}+D_{23})^{\mu_2}\cdots(D_{p2}+D_{p3})^{\mu_p}$$

and

$$\Delta'(D_{13}+D_{12})^{\mu_1'}(D_{22}+D_{23})^{\mu_2'}\cdots(D_{p3}+D_{p2})^{\mu_p'},$$

of which the first ends with $\Delta D_{13}^{\mu_1}\cdots D_{p3}^{\mu_p}$ and the second begins with $\Delta' D_{13}^{\mu_1'}\cdots D_{p3}^{\mu_p'}$ and these are "consecutive" since the original terms were so.

Similarly when $(D_{11}+D_{12}+D_{13})^{\lambda_1}\cdots(D_{p1}+D_{p2}+D_{p3})^{\lambda_p}$ has been expanded so as to satisfy the conditions (i), (ii), (iii) we can deduce the correct expansion of

$$(D_{11}+D_{12}+D_{13}+D_{14})^{\lambda_1}\cdots(D_{p1}+D_{p2}+D_{p3}+D_{p4})^{\lambda_p}$$

as follows:

In the last term in which D_{k3} occurs replace it by $D_{k3} + D_{k4}$,
In the term before replace it by $D_{k4} + D_{k3}$,
In the term before that replace it by $D_{k3} + D_{k4}$,

and so on, and we obtain an ordered expansion starting with $D_{11}^{\lambda_1} D_{21}^{\lambda_2} \cdots D_{p1}^{\lambda_p}$ and ending with $D_{14}^{\lambda_1} D_{24}^{\lambda_2} \cdots D_{p4}^{\lambda_p}$.

It is evident that the method is perfectly general, and we thus succeed in ordering the terms of the transvectant of any two binary forms.

As an illustration we will order the 56 terms of

$$T = \{(ax)^m (bx)^n (cx)^p,\ (dx)^s (ex)^t\}^3.$$

§ 10. Let

$$T = \{(ax)^m (bx)^n (cx)^p,\ (dx)^s (ex)^t\}^3.$$

The polarizing operator is $(D_1 + D_2 + D_3)^3$. Now

$$(D_3 + D_2)^3 = D_3^3 + D_3^2 D_2 + D_2^2 D_3 + D_2^3;$$

$$\therefore\ (D_3 + D_2 + D_1)^3 = D_3^3 + D_3^2 (D_2 + D_1) + D_3 (D_1 + D_2)^2 + (D_2 + D_1)^3,$$

reversing this, we find

$$(D_1 + D_2 + D_3)^3 = D_1^3 + D_1^2 D_2 + D_1 D_2^2 + D_2^3 + D_2^2 D_3 + D_2 D_1 D_3 + D_1^2 D_3 + D_1 D_3^2 + D_2 D_3^2 + D_3^3.$$

Let D_{11} polarize (dx) with regard to y_1	Let D_{12} polarize (ex) with regard to y_1
D_{21} ———— y_2	D_{22} ———— y_2
D_{31} ———— y_3	D_{23} ———— y_3

Also let

$$\begin{array}{rr|rr|rr} y_{11} = & a_2 & y_{21} = & b_2 & y_{31} = & c_2 \\ y_{12} = & -a_1 & y_{22} = & -b_1 & y_{32} = & -c_1 \end{array}$$

Then

$$\begin{aligned} T = {} & (ax)^{m-3} (bx)^n (cx)^p (D_{11} + D_{12})^3 \{(dx)^s (ex)^t\} && 1 \\ & + (ax)^{m-2} (bx)^{n-1} (cx)^p (D_{22} + D_{21})(D_{12} + D_{11})^2 \{(dx)^s (ex)^t\} && 2 \\ & + (ax)^{m-1} (bx)^{n-2} (cx)^p (D_{11} + D_{12})(D_{21} + D_{22})^2 \{(dx)^s (ex)^t\} && 3 \\ & + (ax)^m (bx)^{n-3} (cx)^p (D_{22} + D_{21})^3 \{(dx)^s (ex)^t\} && 4 \\ & + (ax)^m (bx)^{n-2} (cx)^{p-1} (D_{31} + D_{32})(D_{21} + D_{22})^2 \{(dx)^s (ex)^t\} && 5 \\ & + (ax)^{m-1} (bx)^{n-1} (cx)^{p-1} (D_{12} + D_{11})(D_{22} + D_{21})(D_{32} + D_{31}) \{(dx)^s (ex)^t\} && 6 \\ & + (ax)^{m-2} (bx)^n (cx)^{p-1} (D_{31} + D_{32})(D_{11} + D_{12})^2 \{(dx)^s (ex)^t\} && 7 \\ & + (ax)^{m-1} (bx)^n (cx)^{p-2} (D_{12} + D_{11})(D_{32} + D_{31})^2 \{(dx)^s (ex)^t\} && 8 \\ & + (ax)^m (bx)^{n-1} (cx)^{p-2} (D_{21} + D_{22})(D_{31} + D_{32})^2 \{(dx)^s (ex)^t\} && 9 \\ & + (ax)^m (bx)^n (cx)^{p-3} (D_{32} + D_{31})^3 \{(dx)^s (ex)^t\}. && 10 \end{aligned}$$

Thus

$$T = T_1 + T_2 + \cdots + T_9 + T_{10}.$$

The operator in T_1 is $D_{11}^3 + D_{11}^2 D_{12} + D_{11} D_{12}^2 + D_{12}^3$

in T_2 is $D_{12}^2(D_{22}+D_{21}) + D_{11}D_{12}(D_{21}+D_{22}) + D_{11}^2(D_{11}+D_{12})$

in T_3 is $D_{21}^2(D_{11}+D_{12}) + D_{21}D_{22}(D_{12}+D_{11}) + D_{22}^2(D_{11}+D_{12})$

in T_4 is $D_{22}^3 + D_{22}^2 D_{21} + D_{22}D_{21}^2 + D_{21}^3$

in T_5 is $D_{21}^2(D_{31}+D_{32}) + D_{21}D_{22}(D_{32}+D_{31}) + D_{22}^2(D_{31}+D_{32})$

in T_6 is $D_{12}D_{22}D_{32} + D_{12}D_{22}D_{31} + D_{12}D_{21}D_{32} + D_{12}D_{21}D_{31}$
$+ D_{11}D_{22}D_{31} + D_{11}D_{22}D_{32} + D_{11}D_{21}D_{32} + D_{11}D_{21}D_{31}$

in T_7 is $D_{11}^2(D_{31}+D_{32}) + D_{11}D_{12}(D_{32}+D_{31}) + D_{12}^2(D_{31}+D_{32})$

in T_8 is $D_{32}^2(D_{12}+D_{11}) + D_{32}D_{31}(D_{11}+D_{12}) + (D_{12}+D_{11})D_{31}^2$

in T_9 is $D_{31}^2(D_{21}+D_{22}) + D_{31}D_{32}(D_{22}+D_{21}) + D_{32}^2(D_{21}+D_{22})$

in T_{10} is $D_{32}^3 + D_{32}^2 D_{31} + D_{32}D_{31}^2 + D_{31}^3$.

Performing the indicated polarizations and remembering that $(\alpha y_1)_{y_{11}=a_2,\, y_{12}=-a_1}$ is (αa) we easily obtain the following development:

$$T_1 = (ax)^{m-3}(bx)^n(cx)^p\{(dx)^{s-3}(ex)^t(da)^3 + (dx)^{s-2}(ex)^{t-1}(da)^2(ea) + (dx)^{s-1}(ex)^{t-2}(da)(ea)^2 + (dx)^s(ex)^{t-3}(ea)^3\},$$

$$T_2 = (ax)^{m-2}(bx)^{n-1}(cx)^p\{(dx)^s(ex)^{t-3}(eb)(ea)^2 + (dx)^{s-1}(ex)^{t-2}(db)(ea)^2 + (dx)^{s-2}(ex)^{t-1}(da)(db)(ea) + (dx)^{s-1}(ex)^{t-2}(da)(ea)(eb) + (dx)^{s-2}(ex)^{t-1}(eb)(da)^2 + (dx)^{s-3}(ex)^t(db)(da)^2\},$$

$$T_3 = (ax)^{m-1}(bx)^{n-2}(cx)^p\{(dx)^{s-3}(ex)^t(da)(db)^2 + (dx)^{s-2}(ex)^{t-1}(ea)(db)^2 + (dx)^{s-1}(ex)^{t-2}(db)(ea)(eb) + (dx)^{s-2}(ex)^{t-1}(da)(db)(eb) + (dx)^{s-1}(ex)^{t-2}(da)(eb)^2 + (dx)^s(ex)^{t-3}(ea)(eb)^2\},$$

$$T_4 = (ax)^m(bx)^{n-3}(cx)^p\{(dx)^s(ex)^{t-3}(eb)^3 + (dx)^{s-1}(ex)^{t-2}(eb)^2(db) + (dx)^{s-2}(ex)^{t-1}(eb)(db)^2 + (ex)^t(dx)^{s-3}(db)^3\},$$

$$T_5 = (ax)^m(bx)^{n-2}(cx)^{p-1}\{(dx)^{s-3}(ex)^t(dc)(db)^2 + (dx)^{s-2}(ex)^{t-1}(ec)(db)^2 + (dx)^{s-1}(ex)^{t-2}(ec)(eb)(db) + (dx)^{s-2}(ex)^{t-1}(dc)(db)(eb) + (dx)^{s-1}(ex)^{t-2}(dc)(eb)^2 + (dx)^s(ex)^{t-3}(ec)(eb)^2\},$$

$$T_6 = (ax)^{m-1}(bx)^{n-1}(cx)^{p-1}\{(dx)^s(ex)^{t-3}(ea)(eb)(ec) + (dx)^{s-1}(ex)^{t-2}(ea)(eb)(dc) + (dx)^{s-1}(ex)^{t-2}(ea)(db)(ec) + (dx)^{s-2}(ex)^{t-2}(ea)(db)(dc) + (dx)^{s-2}(ex)^{t-1}(da)(eb)(dc) + (dx)^{s-1}(ex)^{t-2}(da)(eb)(ec) + (dx)^{s-2}(ex)^{t-1}(da)(db)(ec) + (dx)^{s-3}(ex)^t(da)(db)(dc)\},$$

$$T_7 = (ax)^{m-2}(bx)^n(cx)^{p-1}\{(dx)^{s-3}(ex)^t(da)^2(dc) + (dx)^{s-2}(ex)^{t-1}(da)^2(ec) + (dx)^{s-1}(ex)^{t-2}(ec)(da)(ea) + (dx)^{s-2}(ex)^{t-1}(dc)(da)(ea) + (dx)^{s-1}(ex)^{t-2}(dc)(ea)^2 + (dx)^s(ex)^{t-3}(ec)(ea)^2\},$$

$$T_8 = (ax)^{m-1}(bx)^n(cx)^{p-2}\{(dx)^s(ex)^{t-3}(ea)(ec)^2 + (dx)^{s-1}(ex)^{t-2}(da)(ec)^2 + (dx)^{s-2}(ex)^{t-1}(da)(ec)(dc) + (dx)^{s-1}(ex)^{t-2}(ea)(ec)(dc) + (dx)^{s-2}(ex)^{t-1}(ea)(dc)^2 + (dx)^{s-3}(ex)^t(da)(dc)^2\},$$

$$T_9 = (ax)^m(bx)^{n-1}(cx)^{p-2}\{(dx)^{s-3}(ex)^t(db)(dc)^2 + (dx)^{s-2}(ex)^{t-1}(eb)(dc)^2 + (dx)^{s-1}(ex)^{t-2}(eb)(dc)(ec) + (dx)^{s-2}(ex)^{t-1}(db)(dc)(ec) + (dx)^{s-1}(ex)^{t-2}(db)(ec)^2 + (dx)^s(ex)^{t-3}(eb)(ec)^2\},$$

$$T_{10} = (ax)^m(bx)^n(cx)^{p-3}\{(dx)^s(ex)^{t-3}(ec)^3 + (dx)^{s-1}(ex)^{t-2}(ec)^2(dc) + (dx)^{s-2}(ex)^{t-1}(ec)(dc)^2 + (dx)^{s-3}(ex)^t(dc)^3\}.$$

These are the 56 terms of $\{(ax)^m(bx)^n(cx)^p,\ (dx)^s(ex)^t\}^3$ so arranged that each term is adjacent to those on either side of it, the numerical coefficients being omitted throughout.

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A CLASS OF NUMBERS CONNECTED WITH PARTITIONS.

By E. T. Bell.

I. The Numbers A.

From their common origin in the elliptic theta and modular functions the theory of partitions and that of the representation of an integer as a sum of squares must be closely related. The innumerable relations thus suggested depend upon eight new systems of integers A which we define in § 1, express in terms of known arithmetical functions and calculate by simple recurrences in § 3. These numbers are functions of two integer parameters, the rank n and the degree r. The numbers of degrees 2, 3, 6, 9 are connected with the class number for binary quadratic forms of a negative determinant and, more generally, those of degrees r, $2r$ (r an arbitrary constant integer > 0) are related to the representation of an integer as a sum of r squares. The connection with partitions is effected through the concepts of the index and degree of a partition introduced in § 1. It will be seen that the subject, which is new, is of great extent.

§ 1. Consider all those partitions of the integer $n > 0$ in which no part appears more than r times. If in a particular one there are precisely a_j parts each of which occurs exactly j times, the r index of the partition is the hypercomplex number $(a_1, a_2, \cdots, a_r)$, and the partition is said to be of degree r.

The number of partitions of n having $(a_1, a_2, \cdots, a_r)$ as index will be written $A_n(a_1, a_2, \cdots, a_r)$, and if $x_1, x_2, \cdots, x_r$ are either constants or functions of a single parameter, we shall write

$$(1) \qquad A_n'(x_1, x_2, \cdots, x_r) \equiv \sum A_n(a_1, a_2, \cdots, a_r)x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r},$$

$\sum$ extending to all $(a_1, a_2, \cdots, a_r)$ for n constant.

When all the parts in each of the partitions enumerated by $A_n(a_1, a_2, \cdots, a_r)$ are restricted to be odd, O will be written in place of A,

$$(2) \qquad O_n'(x_1, x_2, \cdots, x_r) \equiv \sum O_n(a_1, a_2, \cdots, a_r)x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}.$$

The functions (1), (2) have the conventional value 1 when $n = 0$.

Since in any partition of n no part can appear more than n times, n is the maximum degree of any partition of n. Hence when $r = n$ in (1), (2), there is no restriction upon the number of times that any part may occur in any of the indicated partitions.

Observing that the exponents n, $2n$, $3n$, $\cdots$ of q on the left of (3) can be written n, $n+n$, $n+n+n$, $\cdots$, we have

$$(3)\quad \prod_{n=1}^{\infty}(1+x_1q^n+x_2q^{2n}+\cdots+x_rq^{rn})=\sum_{n=0}^{\infty}q^nA'_n(x_1,x_2,\cdots,x_r).$$

As always henceforth it is assumed that such a value of q has been chosen as to render absolutely convergent the product and series. It is easy to show that such a q exists in all cases treated in this paper. Similarly, Π extending to $m = 1, 3, 5, \cdots$, we have

$$(4)\quad \prod(1+x_1q^m+x_2q^{2m}+\cdots+x_rq^{rm})=\sum_{n=0}^{\infty}q^nO'_n(x_1,x_2,\cdots,x_r).$$

There are two distinct divisions of the subject according as $x_1, x_2, \cdots, x_r$ are or are not numerical constants. Here we consider only the important cases in which

$$\begin{aligned} x_s &= \tbinom{r}{s}, & x_s &= (-1)^s\tbinom{r}{s} & (s &= 1, 2, \cdots, r),\\ x_s &= \left\{{r \atop s}\right\}, & x_s &= (-1)^s\left\{{r \atop s}\right\} & (s &= 1, 2, \cdots, n),\end{aligned}$$

where $\binom{r}{s}$ is the binomial coefficient $r!/s!(r-s)!$, $\binom{0}{s} = \binom{0}{0} = 1$, and $\left\{{r \atop s}\right\}$ is the rth figurate number of order s,

$$\left\{{r \atop s}\right\} = \tbinom{r+s-1}{s} = \tbinom{s+r-1}{r-1}.$$

Let n, r denote positive integers. The first class of A numbers comprises the four systems

$$\begin{aligned} A_0(n, r) &\equiv A'_n(-\tbinom{r}{1}, \tbinom{r}{2}, -\tbinom{r}{3}, \cdots, (-1)^r\tbinom{r}{r}),\\ A_1(n, r) &\equiv A'_n(\tbinom{r}{1}, \tbinom{r}{2}, \tbinom{r}{3}, \cdots, \tbinom{r}{r}),\\ A_2(n, r) &\equiv O'_n(\tbinom{r}{1}, \tbinom{r}{2}, \tbinom{r}{3}, \cdots, \tbinom{r}{r}),\\ A_3(n, r) &\equiv O'_n(-\tbinom{r}{1}, \tbinom{r}{2}, -\tbinom{r}{3}, \cdots, (-1)^r\tbinom{r}{r}),\end{aligned}$$

of rank n and degree r, and the second class is

$$\begin{aligned} A_0(n, -r) &\equiv A'_n(\left\{{r \atop 1}\right\}, \left\{{r \atop 2}\right\}, \cdots, \left\{{r \atop n}\right\}),\\ A_1(n, -r) &\equiv A'_n(-\left\{{r \atop 1}\right\}, \left\{{r \atop 2}\right\}, \cdots, (-1)^n\left\{{r \atop n}\right\}),\\ A_2(n, -r) &\equiv O'_n(-\left\{{r \atop 1}\right\}, \left\{{r \atop 2}\right\}, \cdots, (-1)^n\left\{{r \atop n}\right\}),\\ A_3(n, -r) &\equiv O'_n(\left\{{r \atop 1}\right\}, \left\{{r \atop 2}\right\}, \cdots, \left\{{r \atop n}\right\}),\end{aligned}$$

of rank n and degree $-r$. These are definitions; neither set is obtained from the other by changing the sign of r. The reason for the notation appears in a moment. In each of the relevant partitions in $A_j(n, r)$ no part occurs more than r times, while in $A_j(n, -r)$ there is no such restriction.

The product extending to $n = 1, 2, 3, \cdots$, or to $m = 1, 3, 5, \cdots$, we

write in the usual notation

$$(5)\qquad q_0 = \prod(1-q^{2n}),\quad q_1 = \prod(1+q^{2n}),\quad q_2 = \prod(1+q^m),\quad q_3 = \prod(1-q^m),$$

where $q_j \equiv q_j(q)$, and see by (3), (4) that

$$(6)\qquad q_j^{\mp r} = \sum q^{2n} A_j(n, \pm r) \qquad (j = 0, 1),$$

$$(7)\qquad q_j^{\mp r} = \sum q^n A_j(n, \pm r) \qquad (j = 2, 3),$$

$\sum$ extending to $n = 0, 1, 2, \cdots$, and the upper signs or the lower being taken throughout. With (5) we have the well-known identity

$$(8)\qquad q_1 q_2 q_3 = q_1(\sqrt{q})q_3 = 1,$$

from which and (5) we get

$$(9)\qquad A_1(n, s) = (-1)^n A_2(n, -s) = A_3(n, -s) \qquad (s \gtrless 0).$$

We may therefore confine our attention to methods for computing

$$(10)\qquad A_j(n, r) \qquad (j = 0, 1).$$

For easy reference we collect here the expansions of $\vartheta_j = \vartheta_j(q)$:

$$(11)\quad \vartheta_0(-q) = \vartheta_3 = \sum q^{n^2},\quad \vartheta_1'(q^4) = \sum(-1|m)mq^{m^2},\quad \vartheta_2(q^4) = \sum q^{m^2},$$

$\sum$ extending to $n = 0, \pm 1, \pm 2, \cdots$, $m = \pm 1, \pm 3, \pm 5, \cdots$, and $(a|b)$ being the Legendre-Jacobi symbol. We have also

$$(12)\qquad \vartheta_0 = q_0 q_3^2,\qquad \vartheta_1' = 2\sqrt[4]{q}\, q_0^3,\qquad \vartheta_2 = 2\sqrt[4]{q}\, q_0 q_1^2,\qquad \vartheta_3 = q_0 q_2^2,$$

whence, solving for q_j, we get

$$(13)\quad q_0^3 = \vartheta_1'/2\sqrt[4]{q},\ q_2^6 = 2\sqrt[4]{q}\,\vartheta_3^2/\vartheta_0\vartheta_2,\ q_1^6 = \vartheta_2^2/4\sqrt{q}\,\vartheta_0\vartheta_3,\ q_3^6 = 2\sqrt[4]{q}\,\vartheta_0^2/\vartheta_2\vartheta_3,$$

and therefore by the transformation of order 2,

$$(14)\qquad q_1^3 = \vartheta_2/2\sqrt[4]{q}\,\vartheta_0(q^2),\qquad q_3^3 = 2\sqrt[4]{q}\,\vartheta_0/\vartheta_2(\sqrt{q}).$$

§ 2. To illustrate the definitions we verify for $n = 5, 6$ a result proved in § 5:

$$\sum(-1)^{a_1+a_3}3^{a_1+a_2}A_n(a_1, a_2, a_3) = 0 \quad\text{or}\quad (-1|m)m$$

according as n is not or is $(m^2-1)/8$ where $m > 0$ is odd. When $n = 5$ the value of the right is 0, since 41 is not a square; when $n = 6$ we have $m = 7$ and the value of the right is -7. All the partitions of 5 are 5, 41, 32, 311, 221, 2111, 11111, the last of which is of degree 5. The 3 indices of the rest are respectively (1, 0, 0), (2, 0, 0), (2, 0, 0), (1, 1, 0), (1, 1, 0), (1, 0, 1), so that $A_5(2, 0, 0) = A_5(1, 1, 0) = 2$, $A_5(1, 0, 0) = A_5(1, 0, 1) = 1$. Substituting these in the left of the above relation, we find $-3 + 18 - 18$

$+ 3 = 0$. All the partitions of 6 are 6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111, of which the last two are of degrees 4, 6, and the 3 indices of the rest are respectively (1, 0, 0), (2, 0, 0), (2, 0, 0), (1, 1, 0), (0, 1, 0), (3, 0, 0), (1, 0, 1), (0, 0, 1), (0, 2, 0). Hence $A_6(2, 0, 0) = 2$, $A_6(a_1, a_2, a_3) = 1$ for each of the others. Proceeding as before we find $- 3 + 18 - 9 + 3 - 27 + 3 - 1 + 9 = - 7$. The sum on the left of the given identity is $A_0(n, 3)$.

§ 3. Denote by $\zeta_s(n)$, $\zeta'_s(n)$ the sum of the sth powers of all, of the odd, divisors of n. Taking logarithmic derivatives of (6) we find

$$nA_0(n, \pm r) = \mp r \sum \zeta_1(s) A_0(n - s, \pm r), \tag{15}$$

$$nA_1(n, \pm r) = \pm r \sum \zeta'_1(s) A_1(n - s, \pm r), \tag{16}$$

in which $\sum$ (as always henceforth in sums involving s under the sign) extends to all values of $s \geqq 0$, rendering the first arguments in the double argument functions positive. From (15), (16) we can write down the forms of the functions (10) as determinants involving ζ_1, ζ'_1. It is unnecessary to transcribe the results as the recurrences (15), (16), together with (19) to (21) offer a more practicable method of computation.

From $q_j^r q_j^{-r} = 1$ we have

$$A_j(n, - r) = - \sum A_j(s, r) A_j(n - s, - r), \tag{17}$$

and hence the explicit form of the A_j of negative degree in terms of the corresponding A_j of positive degree. We omit the determinant forms. It is therefore sufficient to discuss only the computation of

$$A_j(n, r) \qquad (j = 0, 1), \tag{18}$$

although (15), (16) are at least as useful as (17).

To eliminate all tentative processes from the calculation of (18) we must find recurrences for ζ_1, ζ'_1 in (15), (16). These can be found in many ways; the following is simple. Write $\theta_1(n) \equiv \zeta_1(\tfrac{1}{2}n) + \zeta'_1(\tfrac{1}{2}n)$ or $\zeta_1(n)$ according as n is even or odd, and $\eta_1(n) \equiv 2\zeta'_1(n) - \zeta_1(n)$ Then after some easy reductions of the logarithmic derivatives of the first and third of (12) we find

$$\theta_1(n) - 2\theta_1(n - 1^2) + 2\theta_2(n - 2^2) - \cdots = (-1)^{n-1} n\epsilon(n), \tag{19}$$

where $\epsilon(n) = 0$ or 1 according as n is not or is the square of a positive integer, and

$$\eta_1(n) + \eta_1(n - 1) + \eta_1(n - 3) + \cdots = n\epsilon(8n + 1), \tag{20}$$

the numbers 1, 3, $\cdots$ being triangular. From (19), (20) we compute the

$\theta_1(n)$, $\eta_1(n)$ by rapid recurrences, and hence $\zeta_1(n)$, $\zeta_1'(n)$ from

$$(21) \qquad \zeta_1(n) = \tfrac{1}{3}[2\theta_1(2n) - \eta_1(n)], \qquad \zeta_1'(n) = \tfrac{1}{3}[\theta_1(2n) + \eta_1(n)].$$

The computation of the A numbers has therefore been effected non-tentatively.

II. Relations with Class Numbers.

§ 4. A few will be sufficient. With the usual conventions* let $F(n)$, $F_1(n)$ denote the number of odd, even classes of binary quadratic forms of negative determinant $-n$, and write $D(n) = F(n) - F_1(n)$. Then there are the classical expansions

$$(22) \qquad \begin{aligned} \vartheta_3^3 = 12\sum q^n D(n), \qquad \vartheta_2^3(q^4) &= 8\sum q^{8n+3}F(8n+3), \\ \vartheta_2(q^4)\vartheta_3^2(q^4) &= 4\sum q^{4n+1}F(4n+1), \\ \vartheta_3(q^2)\vartheta_2^2(q^2) &= 4\sum q^{2n+1}F(4n+2), \end{aligned}$$

$\sum$ extending to $n = 0, 1, 2, \cdots$. From (8), (12) we find

$$(23) \quad 2\sqrt[4]{q}\,\vartheta_3^3 = \vartheta_1' q_2^6, \quad \vartheta_2^3 = 4\sqrt{q}\,\vartheta_1' q_1^6, \quad q_3^2\vartheta_2\vartheta_3^2 = \vartheta_1' q_2^2, \quad q_3^2\vartheta_3\vartheta_2^2 = 2\sqrt[4]{q}\,\vartheta_1' q_1^2.$$

Translating these as they stand we get

$$(24) \qquad 12D(n) = \sum (-1)^s(2s+1)A_2(n-s-s^2, 6),$$

$$(25) \qquad F(8n+3) = \sum (-1)^s(2s+1)A_1(n-\tfrac{1}{2}s-\tfrac{1}{2}s^2, 6),$$

$$(26) \quad 2\sum (-1)^s A_2(s, 2)F(4n+1-4s) = \sum (-1)^s(2s+1)A_2(n-s-s^2, 2),$$

$$(27) \quad \sum (-1)^s A_2(s, 2)F(8n+2-4s) = \sum (-1)^s(2s+1)A_1(n-\tfrac{1}{2}s-\tfrac{1}{2}s^2, 2),$$

$$(27.1) \qquad \sum (-1)^s A_2(s, 2)F(8n+6-4s) = 0;$$

while if we divide the first and second of (23) by q_2^6, q_1^6 respectively and apply (9) to the first result we find

$$(28) \qquad \sum (-1)^s A_1(s, 6)D(n-s) = 0 \quad \text{or} \quad m(-1\,|\,m),$$

$$(29) \qquad \sum A_1(n-2s, -6)F(8s+3) = 0 \quad \text{or} \quad m(-1\,|\,m)$$

according as n is not or is $(m^2-1)/4$ where $m > 0$ is odd.

We notice from $\vartheta_3^3 = q_0^3 q_2^6$, $\vartheta_2^3 = 8q_0^3 q_1^6 q^{3/4}$ the following, easily seen to be the same as (24), (25),

$$(30) \qquad 12D(n) = \sum A_0(s, 3)A_2(n-2s, 6),$$

$$(31) \qquad F(8n+3) = \sum A_0(s, 3)A_1(n-s, 6).$$

In any of the above the A_2 can be replaced by their A_1 equivalents by means of (9). See also (37), (38), (49) to (54).

* H. J. S. Smith, "Report on the Theory of Numbers," Art. 135.

III. RELATIONS WITH SQUARE FUNCTIONS.

§ 5. The number of such relations may be multiplied indefinitely. We therefore give only a few representative specimens.

Denote by $N(n, r)$ the number of representations of n as a sum of r squares whose roots are $\gtreqless 0$, and by $M(n, r)$ the like number in which all the squares are odd with roots > 0. Then $M(n, r) = 0$ if $n \not\equiv r \bmod 8$, and we have

$$(32) \qquad \vartheta_3^r = \sum q^n N(n, r), \qquad \vartheta_2^r(q^4) = 2^r \sum q^{8n+r} M(8n + r, r),$$

$\sum$ extending to $n = 0, 1, 2, \cdots$, with the convention that $N(0, r) = 1$.

Interpreted as it stands, the first of the identities (13) gives the result in § 2. Raise each of (14) to the rth power and use (32):

$$(33) \qquad \sum (-1)^s A_1(s, 3r) N(n - s, r) = (-1)^n M(8n + r, r),$$
$$(34) \qquad \sum A_3(s, 3r) M(8n + r - 8s, r) = (-1)^n N(n, r).$$

Raising the last two of (12) to the rth power and taking logarithmic derivatives of the results, we obtain recurrences for M, N:

$$(35) \qquad nM(8n + r, r) = r \sum \eta_1(s) M(8n + r - 8s, r),$$
$$(36) \qquad nN(n, r) = -2r \sum (-1)^s \theta_1(s) N(n - s, r),$$

with which (19), (20) are to be used. These may be solved to give M, N explicitly in terms of η_1, θ_1 if desired.

The special cases $r = 3$ of (33), (34) may be noticed. By (22) we have

$$(37) \qquad 12 \sum (-1)^s A_1(s, 9) D(n - s) = (-1)^s F(8n + 3),$$
$$(38) \qquad \sum A_3(s, 9) F(8n + 3 - 8s) = 12(-1)^n D(n).$$

Another important square function of frequent occurrence is the following. Let $R(n, r, t)$ denote the number of representations of n as a sum of r squares of which precisely t are odd with roots > 0 and occupy the first t places, the roots of the $r - t$ even squares being $\gtreqless 0$ and not fixed as to order. Then

$$(39) \qquad \vartheta_2^r(q^4) \vartheta_3^t(q^4) = 2^r \sum q^{4n+r} R(4n + r, r + t, r).$$

If wished, R can be easily expressed in terms of the corresponding function in which either or both of the signs and positions of the odd squares are free. From (39) or the definitions,

$$(40) \qquad R(4n, t, 0) = N(n, t), \qquad R(8n + r, r, r) = M(8n + r, r);$$

while from the last two of (12),

$$(41) \qquad \sum (-1)^s A_1(s, 2t) R(8n + r - 4s, r + t, r) = \sum A_0(s, r + t) A_1(n - s, 2r),$$

from which, observing that $A_j(0, 0) = 1$, $A_j(n, 0) = 0 (n > 0)$, we have by (40),

$$M(8n + r, r) = \sum A_0(s, r) A_1(n - s, 2r), \tag{42}$$

$$\sum (-1)^s A_1(s, 2r) N(2n - s, r) = A_0(n, r); \tag{43}$$

while again from the last two of (12),

$$\sum (-1)^s A_2(s, 2r) R(4n - 8s + r, r + t, r) = \sum A_0(s, r + t) A_2(n - 2s, 2t), \tag{44}$$

whence by (40),

$$N(n, r) = \sum A_0(s, r) A_2(n - 2s, 2r), \tag{45}$$

$$\sum (-1)^s A_2(s, 2r) = M(8n + r - 8s, r) = A_0(n, r). \tag{46}$$

Some special cases of the above are of particular interest. From (22), (39) we have

$$R(4n + 1, 3, 1) = 4F(4n + 1), \qquad R(4n + 2, 3, 2) = 4F(4n + 2), \tag{47}$$

and therefore on putting $(r, t) = (1, 2), (2, 1)$ in (41), (44) and using the result that

$$A_0(n, 3) = 0 \quad \text{or} \quad m(-1|m) \tag{48}$$

according as n is not or is $(m^2 - 1)/8$ where $m > 0$ is odd, which follows from the first of (13), we find the remarkable relations

$$4\sum (-1)^s A_1(s, 4) F(8n + 1 - 4s) = \sum (-1)^s (2s + 1) A_1(n - \tfrac{1}{2}s - \tfrac{1}{2}s^2, 2), \tag{49}$$

$$4\sum (-1)^s A_1(s, 2) F(8n + 2 - 4s) = \sum (-1)^s (2s + 1) A_1(n - \tfrac{1}{2}s - \tfrac{1}{2}s^2, 4), \tag{50}$$

$$4\sum (-1)^s A_2(s, 2) F(4n + 1 - 8s) = \sum (-1)^s (2s + 1) A_2(n - s - s^2, 4), \tag{51}$$

$$4\sum (-1)^s A_2(s, 4) F(4n + 2 - 8s) = \sum (-1)^s (2s + 1) A_2(n - s - s^2, 2); \tag{52}$$

while in a similar way (42), (45) are seen to be the generalizations of (31), (30) respectively, and from (43), (46) we get*

$$12\sum (-1)^s A_1(s, 6) D(2n - s) = 0 \quad \text{or} \quad m(-1|m), \tag{53}$$

$$\sum (-1)^s A_2(s, 6) F(8n + 3 - 8s) = 0 \quad \text{or} \quad m(-1|m) \tag{54}$$

according as n is not or is $(m^2 - 1)/8$ where $m > 0$ is odd, and (30), (31) give (24), (25).

* The left of (53) is not an integral multiple of 12 since $D(0) = \frac{1}{12}$ by the usual conventions.

§ 6. Another special type deserves notice because most probably it contains only a finite number of distinct relations. It is well known that the number of representations of n as a sum of 2, 4, 6, or 8 squares, and the like for $8n+r$ $(r=2, 4, 6, 8)$ when all the squares are odd, can be expressed in terms of the real divisors of n alone. For certain special forms of n the same holds also for 10, 12 squares, but it is not at present definitely settled, although certain considerations give a strong presumption in favor of its probability, whether these exhaust all such cases. Again it is well known, being implicit in the *Fundamenta Nova* of Jacobi, that $R(4n+r, r+t, r)$ is also so expressible when $(r, t) = (1, 1), (2, 2), (2, 4), (4, 2), (3, 3), (4, 4)$. The appropriate functions of the divisors in all of the above cases are given in convenient form in (among other places) a former paper.* Using these in conjunction with the formulas involving M, N, R, we can readily write out the system of relations between numbers A and functions of divisors. To save space we omit the results.

§ 7. Thus far we have used only the simplest identities from the rudiments of the elliptic theta functions as the point of departure for obtaining relations between the numbers A and square functions, and the ease with which a profusion of results can be so obtained is sufficient evidence of the extent of the theory. We must, however, allude to two further sources of relations, each of which is incomparably more prolific than that which we have used. The first is the theory of transformation and the related modular equations, examples of which are contained in the second, viz., Jacobi's memoir† on infinite series in which the exponents are contained simultaneously in two different quadratic forms. Not attempting here an exhaustive analysis of the relations deduced from Jacobi's expansions, we shall conclude with two examples. In Jacobi's formulas $\sum$ refers to all integers i, k from $-\infty$ to ∞.

As a first example consider (loc. cit., p. 238) Jacobi's result which can be written

$$q^{10}\varphi_0(q^{20})\varphi_0(q^{100}) = \sum (-1)^{i+k} q^{(10i+3)^2+(10k+1)^2},$$

and denote by $U(n, r)$ the sum $\sum(-1)^{i_j+k_j}$ taken over all solutions of $n = \sum_1^r[(10i_j+3)^2+(10k_j+1)^2]$, which we need not define verbally. Then proceeding as before we find

$$\text{(55)}\qquad \sum A_0(s, r)A_0(n-5s, r) = U(40n+10r, r),$$

$$\text{(56)}\qquad A_0(n, r) = \sum A_0(s, -r)U(40n+10r-200s, r),$$

$$\text{(57)}\qquad A_0(n, r) = \sum A_0(s, -r)U(200n+10r-40s, r).$$

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. 42 (1920), p. 168.

† Werke, Vol. 2, pp. 219–288.

When $r = 1$, we have by Euler's theorem $A_0(n, 1) = 0$ or $(-1)^a$ according as n is not or is $(3a^2 + a)/2$ where $a \gtreqless 0$, and evidently $A_0(n, -1)$ is the total number $P(n)$ of partitions of n, also $U(n, 1)$ is the excess of the total number of representations of n in the pair of forms

$$(20i + 3)^2 + (20k + 1)^2, \quad (20i + 13)^2 + (20k + 11)^2 \quad (i, k \gtreqless 0)$$

over the like number for the pair

$$(20i + 13)^2 + (20k + 1)^2, \quad (20i + 3)^2 + (20k + 11)^2 \quad (i, k \gtreqless 0),$$

and from (55) this excess is equal to the excess of the total number of representations of $24n + 6$ in the pair of forms

$$(12i + 1)^2 + (12k + 1)^2, \quad (12i + 7)^2 + (12k + 7)^2 \quad (i, k \gtreqless 0)$$

over the like number for the pair

$$(12i + 7)^2 + (12k + 1)^2, \quad (12i + 1)^2 + (12k + 7)^2 \quad (i, k \gtreqless 0);$$

while from (56), (57) the value of either sum

$$\sum P(s)U(40n + 10 - 200s, 1), \qquad \sum P(s)U(200n + 10 - 40s, 1)$$

is 0 or $(-1)^a$ according as n is not or is $(3a^2 + a)/2$, $a \gtreqless 0$. Conclusions somewhat similar to the first of these can be read off by (48) from (55).

From the modular equation for the transformation of order 7, Jacobi obtains (loc. cit., p. 288) a result equivalent to

$$q^{17}q_2(q^{24})q_0(q^{168}) = q_0^{-1}(q^{48})\sum (-1)^k q^{3(4i+1)^2+14(6k+1)^2};$$

whence we find

$$(58) \quad \sum A_0(s, r)A_2(n - 18s, r) = \sum A_0(s, -r)E(24n + 17 - 96s, r),$$

in which $E(n, r)$ is the excess of the number of representations of n in the form

$$\sum_{i=1}^{r} [3(4a_i + 1)^2 + 14(12b_i + 1)^2] \qquad (a_i, b_i \gtreqless 0)$$

over the like number for

$$\sum_{i=1}^{r} [3(4a_i + 1)^2 + 14(12b_i + 7)^2] \qquad (a_i, b_i \gtreqless 0).$$

When $r = 1$, $E(n, 1)$ is the excess of the number of representations of n in the first of the following forms over the like for the second,

$$3(4a + 1)^2 + 14(12b + 1)^2, \qquad 3(4a + 1)^2 + 14(12b + 7)^2 \qquad (a, b \gtreqless 0),$$

and we have

$$(59) \quad \sum (-1)^a Q(n - 9a - 27a^2) = \sum P(s)E(24n + 17 - 96s, 1),$$

where $Q(n)$ is the number of partitions of n into distinct odd parts and the sum on the left refers to all $a \gtreqless 0$ rendering $n - 9a - 27a^2$ positive.

It is a feature of this subject that all indicated computations can be performed non-tentatively. In particular, all of the arithmetical functions occurring can be calculated by recurrence. For example, taking logarithmic derivatives of the rth power of Jacobi's identity, we find for $E(n, r)$ the recurrence

$$(60) \quad nE(24n + 17r, r) = -\sum \lambda_1(s + 1)E(24n + 17r - 24s - 24, r),$$

where $\lambda_1(n) = -\zeta_1'(n)$ if $n \equiv j \bmod 28$, $j \neq 0, 14$;

$$n \equiv 0 \bmod 28, \qquad \lambda_1(n) = \zeta_1'(n) + 4\zeta_1\left(\frac{n}{4}\right) + 14\zeta_1\left(\frac{n}{14}\right),$$

$$n \equiv 14 \bmod 28, \qquad \lambda_1(n) = \zeta_1'(n) + 14\zeta_1\left(\frac{n}{4}\right),$$

and ζ_1, ζ_1' are to be calculated as in (19)–(21).

December, 1922.

NOTE ON A NEW TYPE OF SUMMABILITY.

By Norbert Wiener.

We have

$$\frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} = \cos x + \frac{\cos 2x}{4} + \cdots + \frac{\cos nx}{n^2} + \cdots. \tag{1}$$

From this it may be deduced at once that

$$\frac{\pi x - x^2}{2} = \sin^2 x + \frac{\sin^2 2x}{4} + \cdots + \frac{\sin^2 nx}{n^2} + \cdots, \tag{2}$$

whence

$$1 - \frac{1}{\pi n} = \frac{2n}{\pi}\left\{ \sin^2\frac{1}{n} + \frac{1}{4}\sin^2\frac{2}{n} + \cdots + \frac{1}{k^2}\sin^2\frac{k}{n} + \cdots \right\}. \tag{3}$$

Moreover,

$$\lim_{n\to\infty} \frac{2n}{\pi k^2}\sin^2\frac{k}{n} = 0. \tag{4}$$

This suggests that given a series $\sum_1^\infty a_n$, we may consider

$$T(a_n) = \lim_{n\to\infty}\frac{2n}{\pi}\left\{ a_1 \sin^2\frac{1}{n} + \frac{a_1 + a_2}{4}\sin^2\frac{2}{n} + \cdots + \frac{a_1 + \cdots + a_k}{k^2}\sin^2\frac{k}{n} + \cdots \right\} \tag{5}$$

as a generalized sum of the a_n's. This definition is of the sort called linear and regular by Carmichael* and Hurwitz,† since (3) and (4) hold, and series (3) is a series of positive terms. Hence it evaluates every convergent series correctly.

Let us now apply our method of summation to the Fourier series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \sum_1^\infty \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n(x - x')dx. \tag{6}$$

The nth partial sum of this is, as is well known,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\frac{\sin (2n+1)\dfrac{x-x'}{2}}{\sin\dfrac{x-x'}{2}}\,dx. \tag{7}$$

* *Bull. Am. Math. Soc.*, Vol. 25 (1918–19), p. 97.

† Ibid., Vol. 28 (1922), p. 20.

Hence we get for (5)

$$\lim_{n\to\infty} \frac{2n}{\pi} \sum_1^\infty \frac{1}{2\pi k^2} \sin^2\frac{k}{n} \int_{-\pi}^{\pi} f(x) \frac{\sin (2n+1)\dfrac{x-x'}{2}}{\sin\dfrac{x-x'}{2}}\, dx. \tag{8}$$

This leads us to consider the series

$$\begin{aligned}
&\frac{2n}{\pi}\sum_{k=1}^\infty \frac{1}{2\pi k^2}\sin^2\frac{k}{n}\frac{\sin (2k+1)u}{\sin u} \\
&= \frac{2n}{\pi}\sum_{k=1}^\infty \frac{1}{4\pi k^2}\left(1-\cos\frac{2k}{n}\right)(\sin 2ku \cot u + \cos 2ku) \\
&= \frac{2n}{\pi}\sum_{k=1}^\infty \frac{1}{4\pi k^2}\left\{\cot u\left[\sin 2ku - \sin 2ku \cos\frac{2k}{n}\right]\right. \\
&\qquad\qquad \left. + \cos 2ku - \cos 2ku \cos\frac{2k}{n}\right\} \\
&= \frac{2n}{\pi}\sum_{k=1}^\infty \frac{1}{4\pi k^2}\left\{\cot u\left[\sin 2ku - \frac{1}{2}\sin 2k\left(u+\frac{1}{n}\right)\right.\right. \\
&\qquad\qquad \left. - \frac{1}{2}\sin 2k\left(u-\frac{1}{n}\right)\right] \\
&\qquad\qquad \left. + \cos 2ku - \frac{1}{2}\cos 2k\left(u+\frac{1}{n}\right) - \frac{1}{2}\cos 2k\left(u-\frac{1}{n}\right)\right\} \qquad (9) \\
&= \frac{n}{4\pi^2}\cot u\Big[-2\int_0^u \log (4\sin^2 x)dx \\
&\qquad + \int_0^{u+(1/n)} \log (4\sin^2 x)dx + \int_0^{u-(1/n)} \log (4\sin^2 x)dx\Big] \\
&\qquad + \frac{n}{4\pi^2}\left[2\left(u^2 - \pi u + \frac{\pi^2}{6}\right) - \left(\left(u+\frac{1}{n}\right)^2\right.\right. \\
&\qquad \left.\left. - \pi\left(u+\frac{1}{n}\right) + \frac{\pi^2}{6}\right) - \left(\left(u-\frac{1}{n}\right)^2 - \pi\left(u-\frac{1}{n}\right)+\frac{\pi^2}{6}\right)\right] \\
&= \frac{n}{4\pi^2}\cot u\Big[\int_u^{u+(1/n)} \log (4\sin^2 x)dx \\
&\qquad\qquad + \int_u^{u-(1/n)} \log (4\sin^2 x)dx\Big] - \frac{1}{2\pi^2 n} \\
&= \frac{1}{4\pi^2}\cot u \log\frac{\sin^2\xi_1}{\sin^2\xi_2} - \frac{1}{2\pi^2 n},
\end{aligned}$$

where ξ_1 lies between u and $u+\dfrac{1}{n}$, ξ_2 between $u-\dfrac{1}{n}$ and u, and $0 < u < \pi$. As n increases, expression (9) converges uniformly to 0 over $(\epsilon, \pi-\epsilon)$. Moreover, in the neighborhood of $u = 0$, expression (9) is positive, while it never has a negative value less than $\dfrac{1}{2\pi^2 n}$. Series (9) is even in u.

The partial sums of series (9) are subject to the formal transformation

$$
\begin{aligned}
&\frac{2n}{\pi}\sum_{1}^{m}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}\frac{\sin(2k+1)u}{\sin u} \\
&\quad=\frac{2n}{\pi}\left\{\frac{1}{2\pi}\sin^2\frac{1}{n}+\frac{1}{\pi}\sin^2\frac{1}{n}\cos 2u\right. \\
&\qquad+\frac{1}{8\pi}\sin^2\frac{2}{n}+\frac{1}{4\pi}\sin^2\frac{2}{n}\cos 2u+\frac{1}{4\pi}\sin^2\frac{2}{n}\cos 4u \\
&\qquad+\frac{1}{18\pi}\sin^2\frac{3}{n}+\frac{1}{9\pi}\sin^2\frac{3}{n}\cos 2u \\
&\qquad\qquad\qquad+\frac{1}{9\pi}\sin^2\frac{3}{n}\cos 4u+\frac{1}{9\pi}\sin^2\frac{3}{n}\cos 6u \qquad (10)\\
&\qquad+\cdots\cdots\cdots\cdots\cdots \\
&\qquad+\frac{1}{2\pi m^2}\sin^2\frac{m}{n}+\frac{1}{\pi m^2}\sin^2\frac{m}{n}\cos 2u \\
&\qquad\qquad\qquad\left.+\cdots+\frac{1}{\pi m^2}\sin^2\frac{m}{n}\cos 2mu\right\} \\
&\quad=\frac{2n}{\pi}\left\{\sum_{1}^{m}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}+\cos 2u\sum_{1}^{m}\frac{1}{\pi k^2}\sin^2\frac{k}{n}\right. \\
&\qquad\qquad\left.+\cos 4u\sum_{2}^{m}\frac{1}{\pi k^2}\sin^2\frac{k}{n}+\cdots+\cos 2mu\frac{1}{\pi m^2}\sin^2\frac{m}{n}\right\}.
\end{aligned}
$$

This suggests an investigation of the series

$$\frac{2n}{\pi}\left\{\sum_{k=1}^{\infty}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}+\sum_{j=1}^{\infty}\cos 2ju\sum_{k=j}^{\infty}\frac{1}{\pi k^2}\sin^2\frac{k}{n}\right\}. \qquad (11)$$

The sum of the squares of the coefficients of (11) may be shown to converge, since the mth coefficient is of the order of $1/m$. Since the coefficients of (10) are positive, less than those of (11) and convergent to those of (11), it follows that the sum of (10) converges in the mean to the sum of (11) with increasing m. Hence series (9) can be integrated term by term when multiplied by any summable function of summable square. Consequently if $f(x)$ is a summable function of summable square, we may write for (8)

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}f(x)\frac{2n}{\pi}\sum_{1}^{\infty}\frac{1}{2\pi k^2}\sin^2\frac{k}{n}\frac{\sin(2k+1)\dfrac{x-x'}{2}}{\sin\dfrac{x-x'}{2}}\,dx. \qquad (12)$$

Let us write (12) in the form $\lim\limits_{n\to\infty}\int_{-\pi}^{\pi}f(x)G_n(x,x')dx$. We have already shown that over $(-\pi, x'-\epsilon)$ and $(x'+\epsilon, \pi)$,

$$\lim_{n\to\infty}G_n(x,x')=0 \qquad (13)$$

uniformly in x, and that $G_n(x, x') + \frac{1}{2\pi^2 n}$ is positive. Moreover,

$$\begin{aligned}\int_{-\pi}^{\pi} G_n(x, x')dx &= \int_{-\pi}^{\pi} \frac{2n}{\pi}\left\{ \sum_1^\infty \frac{1}{2\pi k^2}\sin^2\frac{k}{n} \right. \\ &\qquad \left. + \sum_{j=1}^{\infty} \cos j(x - x') \sum_{k=j}^{\infty} \frac{1}{\pi k^2}\sin^2\frac{k}{n}\right\} dx \\ &= \frac{2n}{\pi}\sum_1^\infty \frac{1}{k^2}\sin^2\frac{k}{n} \\ &= 1 - \frac{1}{\pi n}.\end{aligned} \tag{14}$$

From all these facts it follows that if $f(x)$ is continuous at $x = x'$,

$$\begin{aligned}\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)G_n(x, x')dx &= \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &= \lim_{n\to\infty}\int_{-\pi}^{x'-\epsilon} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &\quad + \lim_{n\to\infty}\int_{x'-\epsilon}^{x'+\epsilon} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &\quad + \lim_{n\to\infty}\int_{x'+\epsilon}^{\pi} f(x)\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &= F\lim_{n\to\infty}\int_{x'-\epsilon}^{x'+\epsilon}\left[G_n(x, x') + \frac{1}{2\pi^2 n}\right]dx \\ &= F\lim_{n\to\infty}\int_{-\pi}^{\pi} G_n(x, x')dx = F,\end{aligned} \tag{15}$$

F being some quantity lying between the upper and the lower bounds of $f(x)$ over the interval $(x' - \epsilon, x' + \epsilon)$. Hence

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)G_n(x, x')dx = f(x'). \tag{16}$$

By an obvious modification of this proof, if $f(x' + 0)$ and $f(x' - 0)$ exist,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)G_n(x, x')dx = \tfrac{1}{2}[f(x' + 0) + f(x' - 0)]. \tag{17}$$

In other words, at any point at which a summable function f of summable square is continuous, its Fourier series may be correctly summed by the method of this paper, and at any point at which f has determinate limits to the right and to the left, its Fourier series is summable to the mean of these values. Since a function is bounded in the neighborhood of a point of continuity, and since the contributions to the partial sums of a Fourier series at a point x' due to values of f for arguments outside $(x' - \epsilon, x' + \epsilon)$ converge uniformly to zero, the condition that f be of summable square is unessential.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November 28, 1922.

ON MEDIATE CARDINALS.

By Dorothy Wrinch.

A "mediate cardinal" is defined in "Principia Mathematica" as a cardinal which is neither inductive nor reflexive and it is established $*(124 \cdot 61)$ that the multiplicative axiom implies the non-existence of mediate cardinals. The converse implication is not established, and there seems to be no reason to suppose it is true. The relation of the existence of mediate cardinals to the multiplicative axiom is therefore one-sided and offers a contrast to the mutual implications of the comparability of cardinals, the well-orderability of classes and the multiplicative axiom. In this paper it is proposed to investigate other classes of cardinals which are not Alephs, beyond the mediate cardinals of "Principia Mathematica," and instead of the one-sided implication between the multiplicative axiom and the non-existence of mediate cardinals to establish an equivalence between the axiom and the non-existence of certain cardinals which are not Alephs.

If we use the term "comparable" in such a sense that μ is comparable with ν when μ is greater than, equal to, or less than ν, a mediate cardinal in "Principia Mathematica" is a cardinal comparable with the inductive cardinals but not with $\aleph_0$.* We wish to discuss the nature of those cardinals (if there are any such) which are comparable with Alephs less than $\aleph_\xi$, but not with $\aleph_\xi$, for different values of ξ. In "Principia Mathematica" it is found that a part of the multiplicative axiom is sufficient to imply the non-existence of the non-inductive non-reflexive cardinals and instead of the proposition

$$\text{Mult Ax} \,.\supset.\, NC\,\text{med} = \Lambda$$

there is the proposition $*124 \cdot 56$

$$\aleph_0 \in NC\,\text{mult} \,.\supset.\, NC\,\text{med} = \Lambda.$$

It seems worth while to pursue the same course with the cardinals to be considered in this paper. For this purpose, it is necessary to establish certain propositions of little interest in themselves. This is done in $*1$.

$*1.$

κ and λ are similar classes of mutually exclusive similar classes. There exists therefore a correlator S of κ and λ, such that if $\alpha \in \lambda$, $S`\alpha \; sm \; \alpha$. We

* $\aleph$ has been substituted for the Hebrew Character Aleph usually employed.

will put

$$Crp(S)‘\alpha = S‘\alpha\,\overline{sm}\,\alpha, \qquad \text{Df.}$$

as in "Principia Mathematica," $*111 \cdot 02$. Then $Crp(S)‘\alpha$ consists of the class of correlators between α and its S-correlate. It is established that under these circumstances

$$R \in \epsilon_\Delta‘Crp(S)``\lambda \,.\supset.\, \dot{s}‘D‘R \in (s‘\kappa)\overline{sm}(s‘\lambda),$$

so that if an R can be found which consists of a selection from the various classes $Crp(S)‘\alpha$ where $\alpha \in \lambda$, $\dot{s}‘D‘R$ is a correlator of $s‘\kappa$ with $s‘\lambda$. It follows that if a selection from the class of classes $Crp(S)``\lambda$ exists, $s‘\kappa$ and $s‘\lambda$ are similar. And this is certainly the case if $Nc‘\lambda \in NC$ mult. Hence we get the proposition

$$*1 \cdot 1 \quad \vdash :. Nc‘\lambda \in NC \text{ mult} \,.\supset:\, \kappa, \lambda \in Cls^2 \text{ excl.} \; \exists ! \,\kappa\,\overline{sm}\,\lambda \cap Rl‘sm \,.\supset.\, s‘\kappa\, sm\, s‘\lambda.$$

κ is a class consisting of $\mathfrak{A}_\xi$ classes of mutually exclusive classes, each having $\mathfrak{A}_\eta$ members. Let α be an $\mathfrak{A}_\eta$ and β an $\mathfrak{A}_\xi$. $\alpha \downarrow ;``\beta$ and κ will then be similar classes of similar mutually exclusive classes. It follows from $1 \cdot 1$ that

$$Nc‘\kappa \in NC \text{ mult} \,.\supset:\, \beta \in \mathfrak{A}_\xi \,.\, \alpha \in \mathfrak{A}_\eta \,.\, \kappa \in Cls^2 \text{ excl} \cap \mathfrak{A}_\xi \cap Cl‘\mathfrak{A}_\eta \,.\supset.\, s‘\kappa\, sm\, s‘\alpha \downarrow ;``\beta.$$

Since $s‘\alpha \downarrow ;``\beta = \beta \times \alpha$ and $Nc‘(\alpha \times \beta) = Nc‘(\beta \times \alpha)$ [P.M. $**113 \cdot 02 \cdot 141$] we get the proposition

$$*1 \cdot 2 \quad \vdash :. \mathfrak{A}_\xi \in NC \text{ mult} \,.\supset:\, \kappa \in Cls^2 \text{ excl} \cap \mathfrak{A}_\xi \cap Cl‘\mathfrak{A}_\eta \,.\supset.\, s‘\kappa \in \mathfrak{A}_\eta \times \mathfrak{A}_\xi.$$

μ and ν are similar classes of mutually exclusive classes. The cardinal number of any member of μ is less than or equal to the cardinal of any member of ν. Then $Nc‘s‘\mu$ is less than or equal to $Nc‘s‘\nu$.

For there exists an S correlating members of μ with members of ν. If $\beta \in \nu$, there exists a subclass of β having the same cardinal as $S‘\beta$, the member of μ with which S correlates β. Putting

$$CN(S)‘\beta = Cl‘\beta \cap Nc‘S‘\beta, \qquad \text{Df.}$$

we have

$$\beta \in \nu \,.\supset.\, \exists ! CN‘\beta.$$

And there are $Nc‘\mu$ β's and $Nc‘\mu$ $CN(S)‘\beta$'s. If $Nc‘\mu \in NC$ mult, a selection ν' can be taken from $CN``\nu$. Then

$$\exists ! \mu\, sm\, \nu' \cap Cl‘sm.$$

Hence by $1 \cdot 1$, if $Nc`\mu \,\epsilon\, NC$ mult

$$s`\mu \,\overline{sm}\, s`\nu'.$$

Since $s`\nu' \subset s`\nu$ it follows that $Nc`s`\mu \leq Nc`s`\nu$. Hence

$*1 \cdot 3 \quad \vdash :: Nc`\mu \,\epsilon\, NC \text{ mult} . \supset :. \mu, \nu \,\epsilon\, Cls^2 \text{ excl. } \mu sm\nu : \alpha \,\epsilon\, \mu \cdot \beta \,\epsilon\, \nu$
$\qquad . \supset_{\alpha\beta} . Nc`\alpha \leq Nc`\beta : \supset . Nc`s`\mu \leq Nc`s`\nu.$

It follows immediately from $1 \cdot 3$, $1 \cdot 2$, putting $\nu = \gamma \updownarrow ``\delta$ where $\gamma \,\epsilon\, \aleph_\eta$, $\delta \,\epsilon\, \aleph_\xi$,

$*1 \cdot 4 \quad \vdash :: Nc`\mu \,\epsilon\, NC \text{ mult} . \supset :. \mu \,\epsilon\, Cls^2 \text{ excl.} \cap \aleph_\xi : \alpha \,\epsilon\, \mu . \supset_\alpha . Nc`\alpha$
$\qquad < \aleph_\eta : \supset . Nc`s`\mu \leq \aleph_\xi \times \aleph_\eta.$

By $1 \cdot 3$, if $\aleph_\zeta \,\epsilon\, NC$ mult, an $\aleph_\xi \times \aleph_\zeta$, where $\xi < \zeta$, is less than or equal to a class which is an $\aleph_\zeta \times \aleph_\zeta$, since it consists of $\aleph_\xi$ classes all of which have less than $\aleph_\zeta$ members. And

$$\aleph_\zeta \times \aleph_\zeta = \aleph_\zeta. \qquad (1 \cdot 41)^*$$

Hence

$$\aleph_\zeta \times \aleph_\xi \leq \aleph_\zeta. \qquad [\text{P.M. } *117 \cdot 6]$$

But

$$\aleph_\zeta \times \aleph_\xi \geq \aleph_\zeta.$$

Therefore

$$\aleph_\zeta \times \aleph_\xi = \aleph_\zeta,$$

which with $1 \cdot 41$ gives

$*1 \cdot 5 \qquad \vdash :. \xi \leq \zeta \cdot \aleph_\zeta \,\epsilon\, NC \text{ mult} . \supset : \aleph_\zeta \times \aleph_\xi = \aleph_\zeta.$

κ is a class of mutually exclusive classes no two of which are similar. The cardinals of the members of κ are

$$\aleph_{\eta+a_0},\ \aleph_{\eta+a_1},\ \cdots\ \aleph_{\eta+a_\zeta},\ \cdots, \qquad (\zeta < \omega_\xi)$$

where the Alephs form an ω_ξ-series, having $\aleph_z$ as limit. κ then consists of $\aleph_\xi$ classes each with less than $\aleph_z$ members. By $1 \cdot 4$

$$\aleph_\xi \,\epsilon\, NC \text{ mult} . \supset . Nc`s`\kappa \leq \aleph_z \times \aleph_\xi.$$

By $1 \cdot 5$ since $\xi < z$

$$\aleph_z \,\epsilon\, NC \text{ mult} . \supset . Nc`s`\kappa \leq \aleph_z.$$

Therefore if P is the series of Alephs in order of magnitude and $\Sigma_c`\lambda$ the arithmetic sum of a class of cardinals λ,

$*1 \cdot 6 \quad \vdash : \lambda \,\epsilon\, Cl`(NC \cap C``\Omega) \cdot lt_P`\lambda \,\epsilon\, NC \text{ mult.} \supset . \Sigma_c`\lambda \leq lt_P`\lambda.$

λ is a class of Alephs whose limit in order of magnitude is $\aleph_z$. Suppose

* See Jourdain, "The Multiplication of Alephs," *Mathematische Annalen*, Bd. LXV, pp. 506–512.

$(\exists x)$. $\Sigma_c`\lambda = x \,.\, x < \aleph_z$. Since $\aleph_z = lt_P`\lambda$

$$(\exists \eta).\ \ x < \eta.\ \ \Sigma_c` \geq \eta.\ \ \eta < \aleph_z.$$

Hence

$$(\exists x).\ \ \Sigma_c`\lambda = x.\ \ x < \aleph_z \,.\, \supset \,.\, \Sigma_c`\lambda > x.$$

Therefore $\sim (\exists x)$. $x < \aleph_z$. $\Sigma_c`\lambda = x$. Hence with 1·6 we get
$*1{\cdot}7 \quad \vdash : \lambda \in Cl`(NC \frown C``\Omega) . \ lt_P`\lambda \in NC \text{ mult} \,.\, \supset \,.\, \Sigma_c`\lambda = lt_P`\lambda$.

$*2$.

We will make use of the definition in "Principia Mathematica,"

$$\text{spec } `\mu = \hat{\nu}[\nu < \mu \,.\, v \,.\, \nu \geq \mu]. \qquad [\text{P.M.}\ *120{\cdot}43]$$

Extending the use of the words "mediate cardinals" to cover cardinals which are comparable with Alephs up to a certain Aleph, instead of using it for non-inductive non-reflexive cardinals, we have mediate cardinals of various degrees.

$$\text{med } `\mu = \hat{\nu}[\rho < \mu \,.\, \supset_\rho \,.\, \nu > \rho \,.\, \sim (\nu \geq \mu \,.\, v \,.\, \nu < \mu)] \qquad \text{Df.}$$

and

$$NC \text{ med} = s` \text{ med } ``NC. \qquad \text{Df.}$$

Then

$$\text{spec } `\aleph_0 = NC \text{ ind } \smile NC \text{ refl},$$
$$\text{med } `\aleph_0 = NC - NC \text{ ind} - NC \text{ refl}$$

so that the non-inductive non-reflexive cardinals are a particular case of classes of mediate cardinals. We have

$$\text{spec } `\aleph_0 \smile \text{med } `\aleph_0 = NC, \qquad 2{\cdot}01$$
$$\text{spec } `\aleph_0 \frown \text{med } `\aleph_0 = \Lambda. \qquad 2{\cdot}011$$

Also

$$\text{spec } `\aleph_1 \smile (\text{med } `\aleph_0 \smile \text{med } `\aleph_1) = NC, \qquad 2{\cdot}02$$
$$\text{spec } `\aleph_1 \frown (\text{med } `\aleph_0 \smile \text{med } `\aleph_1) = \Lambda, \qquad 2{\cdot}021$$

and generally

$$\text{spec } `\aleph_\nu \smile (\text{med } `\aleph_0 \smile \text{med } `\aleph_1 \cdots \smile \text{med } `\aleph_\nu) = NC, \qquad 2{\cdot}03$$
$$\text{spec } `\aleph_\nu \frown (\text{med } `\aleph_0 \smile \text{med } `\aleph_1 \cdots \smile \text{med } `\aleph_\nu) = \Lambda. \qquad 2{\cdot}03'$$

Further

$$\text{med } `\aleph_\nu \frown \text{med } `\aleph_{\nu'} = \Lambda$$

unless $\nu = \nu'$ and spec $`\mu \subset$ spec $`\nu$ if $\mu \geq \nu$. Hence spec $`\nu \doteq p`$ spec $``\overrightarrow{\leqq}``\aleph_\nu$. Therefore from 2·03·03′ it follows that

$*2{\cdot}1 \qquad \vdash . \ NC - \text{spec } `\aleph_\xi = s` \text{ med } ``\overrightarrow{\leqq}``\aleph_\xi$.

From 2.01.02 it follows that

$$(\text{spec } `\aleph_0 \frown \text{spec } `\aleph_1) \smile (\text{med } `\aleph_0 \smile \text{med } `\aleph_1) = NC \qquad 2{\cdot}11$$

and generally from 2·11,

$$p\text{‘ spec ‘‘}\gtrsim\text{‘}\aleph_\xi \cup s\text{‘ spec ‘‘}\lesssim\text{‘}\aleph_\xi = NC \qquad 2\cdot 12$$

and

*2·2 $\quad \vdash . NC - p\text{‘ spec ‘‘}\gtrsim\text{‘}\aleph_\xi = s\text{‘ med ‘‘}\lesssim\text{‘}\aleph_\xi.$

Then from 2·1·04

$$s\text{‘}(NC - \text{spec ‘‘}\lesssim\text{‘}\aleph_\xi) = s\text{‘ med ‘‘}\lesssim\text{‘}\aleph_\xi$$
$$= s\text{‘ med ‘‘}\lesssim\text{‘}\aleph_\xi - \text{med ‘}\aleph_\xi.$$

Thus

*2·3 $\quad \vdash . s\text{‘}(NC - \text{spec ‘‘}\lesssim\text{‘‘}\aleph_\xi) = NC - \text{spec ‘}\aleph_\xi - \text{med ‘}\aleph_\xi.$

From 2·1 it follows that

$$\text{med ‘}\aleph_\xi \cup \text{spec ‘}\aleph_\xi \cup s\text{‘ med ‘‘}\gtrsim\text{‘}\aleph_\xi = NC.$$

Therefore from 2·2 we get

*2·4 $\quad \vdash . \text{med ‘}\aleph_\xi \cup \text{spec ‘}\aleph_\xi = p\text{‘ spec ‘‘}\gtrsim\text{‘}\aleph_\xi.$

Taking 2·4 in the particular case when $\xi = \zeta + 1$, we get

*2·5 $\quad \vdash . \text{med ‘}\aleph_{\zeta+1} \cup \text{spec ‘}\aleph_{\zeta+1} = p\text{‘ spec ‘‘}\gtrsim\text{‘}\aleph_\zeta.$

Further we get the proposition

*2·6 $\quad \vdash : \aleph_z \in NC \text{ mult. } \sim(\exists\zeta\cdot\zeta + 1 = z) . \supset . \text{med ‘}\aleph_z = \Lambda.$

For suppose α is a class containing subclasses having as cardinals all Alephs less than $\aleph_z$. There will be a well-ordered series of Alephs

$$\aleph_{\eta+\alpha_0},\quad \aleph_{\eta+\alpha_1},\quad \cdots \quad \aleph_{\eta+\alpha_\omega},\quad \cdots$$

with $\aleph_z$ as limit. And each of the classes

$$Cl\text{‘}\alpha \cap \aleph_{\eta+\alpha_0};\quad Cl\text{‘}\alpha \cap \aleph_{\eta+\alpha_1};\quad \cdots \quad Cl\text{‘}\alpha \cap \aleph_{\eta+\alpha_\omega};$$

will have at least one member. The number of these classes is less than $\aleph_z$. Hence, if $\aleph_z \in NC$ mult, a selection κ can be made from this class of classes. From 1·7 it follows that $s\text{‘}\kappa$ will be an $\aleph_z$. It is therefore clear that if a class contains subclasses having as cardinals all Alephs less than $\aleph_z$, it also contains an $\aleph_z$. There can therefore be no med ‘$\aleph_z$.

From **2·1·6 it follows that

*2·7 $\quad \vdash : \aleph_z \in NC \text{ mult} . \sim(\exists\zeta\cdot\zeta + 1 = z) . \supset . NC - \text{spec ‘}\aleph_z$
$$= s\text{‘ med ‘‘}\lesssim\text{‘}\aleph_z.$$

$*3.$

It has been established that the following assumptions are equivalent *inter se.*

$*3\cdot 1$ The Multiplicative Axiom.

$*3\cdot 2$ The Comparability of Cardinals.

$*3\cdot 3$ The Well-Orderability of Classes.*

It was also established that the multiplicative axiom implies the non-existence of med '$\aleph_0$. It was not proved and there seems no reason to suppose it is true that the non-existence of med '$\aleph_0$ implies the multiplicative axiom. The consideration of the other mediate cardinals, however, shows that the truth of the multiplicative axiom would be implied by the non-existence of med '$\aleph_0$ together with the non-existence of the mediate cardinals corresponding to Alephs other than $\aleph_0$. For we have

$$NC = \text{Aleph} . \supset : \mu \in NC . \supset . \mu \in p\text{`}\,\text{spec}\text{``}\text{Aleph}.$$

And therefore

$$NC = \text{Aleph} . \supset . NC = p\text{`}\,\text{spec}\text{``}\text{Aleph}. \qquad (3\cdot 31)$$

Since

$$\begin{aligned} NC = \text{Aleph} . \supset &: \mu \in NC . \supset_\mu . \mu \in \text{Aleph} \\ \supset &: \mu \in NC . \supset_\mu . \exists \nu . \nu \in \text{Aleph} . \mu \in \text{spec}\text{`}\,\nu, \end{aligned}$$

therefore

$$NC = \text{Aleph} . \supset . NC = s\text{`}\,\text{spec}\text{``}NC,$$

which with $3\cdot 31$ gives

$$NC = \text{Aleph} . \supset . NC = s\text{`}\,\text{spec}\text{``}NC = p\text{`}\,\text{spec}\text{``}NC.$$

Further

$$NC = p\text{`}\,\text{spec}\text{``}NC . \supset : \mu \in NC \cdot \nu \in NC . \supset . \mu \geq \nu . \vee . \mu < \nu.$$

This with the fact of the equivalence of $3\cdot 2$ and $3\cdot 3$ gives a fourth assumption which is equivalent to $3\cdot 1$, $3\cdot 2$ and $3\cdot 3$, viz.,

$*3\cdot 4$ $$NC = p\text{`}\,\text{spec}\text{``}NC = s\text{`}\,\text{spec}\text{``}NC.$$

It follows that $\mu \in NC . \supset . NC = \text{spec}\text{`}\,\mu$. Then from $2\cdot 1$ it can be deduced that

$$\begin{aligned} Cls = 1 \cup C\text{``}\Omega . \supset . s\text{`}\,\text{med}\text{``}\text{Aleph} &= \Lambda, \\ Cls = 1 \cup Cl\text{`}\Omega . \supset . s\text{`}\,\text{med}\text{``}NC \quad &= \Lambda. \end{aligned}$$

Finally from $2\cdot 1$, if $s\text{`}\,\text{med}\text{``}\text{Aleph} = \Lambda$, $\mu \in \text{Aleph} . \supset . NC = \text{spec}\text{`}\mu$. Hence $NC = p\text{`}\,\text{spec}\text{``}\text{Aleph} = s\text{`}\,\text{spec}\text{``}\text{Aleph}$. If $\mu \in NC - \text{Aleph}$, then there is some Aleph to whose species $\mu$ does not belong and therefore $NC \neq p\text{`}\,\text{spec}\text{`}\text{Aleph}$. Hence if $s\text{`}\,\text{med}\text{``}NC = \Lambda$, there are no cardinals not Alephs. Thus we have established the equivalence of the statements $3\cdot 1$, $3\cdot 2$ and $3\cdot 3$ and $3\cdot 4$ and the statement

$*3\cdot 5$ $$NC\,\text{med} = \Lambda.$$

* See "Principia Mathematica," $**258\cdot 37\cdot 39$, and Hartogs, "Über das Problem der Wohlordnung," *Mathematische Annalen*, Bd. LXXVI.

PERIODIC OSCILLATIONS OF THREE FINITE MASSES ABOUT THE LAGRANGIAN CIRCULAR SOLUTIONS.

By H. E. Buchanan.

Introduction.—In 1772 LaGrange announced the particular solutions of the problem of three bodies in which the ratios of the mutual distances remain constant. These are the so-called straight line and equilateral triangle solutions. Liouville, in 1845,* investigated the stability of these solutions and found that for a small displacement the bodies would, in general, depart to relatively great distances from the starting place. The question still remained whether or not these displacements could be selected so that a set of periodic orbits would result. In this paper several classes of such orbits in the vicinity of the straight line solutions are shown to exist. In case one of three bodies is infinitesimal the problem has been treated by Moulton in Chapter V of his Periodic Orbits. The method which was developed there will be applied here when all of the bodies are finite.

The Differential Equations.—The general differential equations of motion for three finite bodies whose coördinates are referred to axes rotating uniformly in the $\xi\eta$-plane are

$$\left.\begin{aligned} \frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} &= \omega^2\xi_i - \sum_{j=1}^{3} m_j \frac{(\xi_i - \xi_j)}{r_{ij}^3} = \frac{1}{m_i}\frac{\partial U}{\partial \xi_i}, \\ \frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} &= \omega^2\eta_i - \sum_{j=1}^{3} m_j \frac{(\eta_i - \eta_j)}{r_{ij}^3} = \frac{1}{m_i}\frac{\partial U}{\partial \eta_i}, \\ \frac{d^2\zeta_i}{dt^2} &= -\sum_{j=1}^{3} m_j \frac{(\zeta_i - \zeta_j)}{r_{ij}^3} = \frac{1}{m_i}\frac{\partial U}{\partial \zeta_i}, \\ U &= \tfrac{1}{2}\sum_{i=1}^{3}\omega^2 m_i(\xi_i^2 + \eta_i^2) + \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{m_i m_j}{r_{ij}} \quad (j \neq i), \end{aligned}\right\} \quad (1)$$

where ω denotes the angular speed, m_i one of the masses, and where the units have been chosen so that the Gaussian constant is unity.

Let the coördinates in the straight line solutions be

$$\xi_i = \xi_i^{(0)}, \quad \eta_i = \zeta_i = 0 \quad (i = 1, 2, 3),$$

* "Connaissance des Temps," 1845.

and make the transformation

$$\xi_i = \xi_i^{(0)} + x'_i, \qquad \eta_i = y'_i, \qquad \zeta_i = z'_i;$$

then equations (1) become

$$\left.\begin{aligned}
&\frac{d^2x'_i}{dt^2} - 2\omega\frac{dy'_i}{dt} = \omega^2(x'_i + \xi_i^{(0)}) - \sum_{j=1}^{3} m_j\frac{(x'_i + \xi_i^{(0)} - x'_j - \xi_j^{(0)})}{r_{ij}^3},\\
&\frac{d^2y'_i}{dt^2} + 2\omega\frac{dx'_i}{dt} = \omega^2 y'_i - \sum_{j=1}^{3} m_j\frac{(y'_i - y'_j)}{r_{ij}^3},\\
&\qquad\frac{d^2z'_i}{dt^2} = -\sum_{j=1}^{3} m_j\frac{(z'_i - z'_j)}{r_{ij}^3} \quad (i = 1, 2, 3;\ j \neq i).
\end{aligned}\right\} \tag{2}$$

These are the equations which define the oscillations.

Expansion of the Right Members.—The right members of equations (2) will be expanded as a power series in x'_i, y'_i, z'_i. The region of convergence is the common region of convergence of the expansions of

$$\frac{1}{r_{12}} = \frac{1}{\sqrt{(x'_2 + \xi_2^{(0)} - x'_1 - \xi_1^{(0)})^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}},$$

$$\frac{1}{r_{13}} = \frac{1}{\sqrt{(x'_3 + \xi_3^{(0)} - x'_1 - \xi_1^{(0)})^2 + (y'_3 - y'_1)^2 + (z'_3 - z'_1)^2}},$$

$$\frac{1}{r_{23}} = \frac{1}{\sqrt{(x'_3 + \xi_3^{(0)} - x'_2 - \xi_2^{(0)})^2 + (y'_3 - y'_2)^2 + (z'_3 - z'_2)^2}}.$$

The first equation can be written in the form

$$\frac{1}{r_{12}} = \frac{1}{x'_2 + \xi_2^{(0)} - x'_1 - \xi_1^{(0)}}\,\frac{1}{\sqrt{1 + \dfrac{(y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}{(x'_2 + \xi_2^{(0)} - x'_1 - \xi_1^{(0)})^2}}},$$

with similar expressions for $1/r_{13}$ and $1/r_{23}$. It follows that the expansions will converge if

$$\frac{(y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}{(x'_2 + \xi_2^{(0)} - x'_1 - \xi_1^{(0)})^2} < +1, \qquad -1 < \frac{x'_2 - x'_1}{\xi_2^{(0)} - \xi_1^{(0)}} < +1,$$

$$\frac{(y'_3 - y'_1)^2 + (z'_3 - z'_1)^2}{(x'_3 + \xi_3^{(0)} - x'_1 + \xi_1^{(0)})^2} < +1, \qquad -1 < \frac{x'_3 - x'_1}{\xi_3^{(0)} - \xi_1^{(0)}} < +1,$$

$$\frac{(y'_3 - y'_2)^2 + (z'_3 - z'_2)^2}{(x'_3 + \xi_3^{(0)} - x'_2 - \xi_2^{(0)})^2} < +1, \qquad -1 < \frac{x'_3 - x'_2}{\xi_3^{(0)} - \xi_2^{(0)}} < +1.$$

The three conditions on the left are satisfied if the line joining any two of the bodies always makes an angle of less than 45° with the x-axis.

Inside the regions defined by these inequalities the right members can be expanded as power series in x'_i, y'_i, z'_i. Then the differential equations can be written in the form

$$\left.\begin{aligned} \frac{d^2x'_i}{dt^2} - 2\omega\frac{dy'_i}{dt} &= X_{1i} + X_{2i} + \cdots, \\ \frac{d^2y'_i}{dt^2} + 2\omega\frac{dx'_i}{dt} &= Y_{1i} + Y_{2i} + \cdots, \\ \frac{d^2z'_i}{dt^2} &= Z_{1i} + Z_{2i} + \cdots \quad (i = 1, 2, 3), \end{aligned}\right\} \tag{3}$$

where the X_{ji}, Y_{ji}, Z_{ji} are homogeneous functions of x'_i, y'_i, z'_i of degree j. Further X_{ji} is a function of y'^2_i and z'^2_i; Y_{ji} is y'_i times a function of y'^2_i and z'^2_i; Z_{ji} is z'_i times a function of y'^2_i and z'^2_i.

THE SYMMETRY THEOREM.—By making use of the properties of X_{ji}, Y_{ji} and Z_{ji} we will prove the theorem:

If all the bodies are projected from and at right angles to the x-axis, the orbits will be symmetrical with respect to this axis geometrically and in the time.

If the initial projection is from and at right angles to the x-axis, then

$$x'_i(0) = a_i, \qquad \frac{dx'_i}{dt}(0) = 0, \qquad y'_i(0) = 0, \qquad \frac{dy'_i}{dt}(0) = b_i, \qquad z'_i(0) = 0,$$
$$\frac{dz'_i}{dt}(0) = c_i,$$

and the solutions may be written

$$\begin{aligned} x'_i &= \varphi_i(a_i, b_i, c_i, t), \\ y'_i &= \psi(a_i, b_i, c_i, t), \\ z'_i &= \theta(a_i, b_i, c_i, t) \quad (i = 1, 2, 3). \end{aligned}$$

Equations (3) remain unchanged when t is replaced by $-t$; y'_i by $-y'_i$ and z'_i by $-z'_i$; but the solutions for the same initial conditions become

$$\begin{aligned} x'_i &= \quad \varphi_i(a_i, b_i, c_i, -t), \\ y'_i &= -\psi_i(a_i, b_i, c_i, -t), \\ z'_i &= -\theta_i(a_i, b_i, c_i, -t). \end{aligned}$$

The φ_i are therefore even functions of t while the ψ_i and the θ_i are odd functions of t.

The Parameters ϵ and δ.—Applying to equations (3) the transformation

$$x'_i = \epsilon' x_i, \qquad y'_i = \epsilon' y_i, \qquad z'_i = \epsilon' z_i, \qquad t = (1 + \delta)\tau,$$

we get

$$\left.\begin{aligned}\frac{d^2x_i}{d\tau^2}-2\omega(1+\delta)\frac{dy_i}{d\tau}&=(1+\delta)^2\{X_{1i}+\epsilon' X_{2i}+\cdots\},\\ \frac{d^2y_i}{d\tau^2}+2\omega(1+\delta)\frac{dx_i}{d\tau}&=(1+\delta)^2\{Y_{1i}+\epsilon' Y_{2i}+\cdots\},\\ \frac{d^2z_i}{d\tau^2}&=(1+\delta)^2\{Z_{1i}+\epsilon' Z_{2i}+\cdots\}\quad(i=1,2,3).\end{aligned}\right\}\tag{4}$$

These equations are valid for the physical problem proposed so long as the bodies remain in the regions of convergence previously determined. Let us generalize the problem by replacing ϵ' by ϵ, a parameter which can take all values in the neighborhood of zero. We get a solution of the physical problem only when $\epsilon' = \epsilon$. The method will be to find all periodic solutions when $\epsilon = \delta = 0$ and discuss the analytic continuation of these when ϵ increases to the value ϵ'.

The Center of Gravity Integrals and the Characteristic Exponents.— Equations (4) possess six integrals which define the position and motion of the center of gravity. When the origin is at the center of gravity, as it is in equations (4), these integrals are

$$\begin{gathered}m_1x_1+m_2x_2+m_3x_3=0,\qquad m_1y_1+m_2y_2+m_3y_3=0,\\ m_1z_1+m_2z_2+m_3z_3=0\end{gathered}\tag{5}$$

and their first derivatives. By means of these equations it is possible to eliminate three of the nine equations of (4). We choose to eliminate the x_2, y_2 and z_2 equations, so that hereafter i takes only the values 1 and 3.

To obtain the generating solutions we put $\epsilon = \delta = 0$ in equations (4). Evidently all except the first terms of the right members disappear. If we remember that x_2, y_2 and z_2 have been eliminated by (5), we obtain the linear homogeneous differential equations

$$\left.\begin{aligned}\frac{d^2x_1}{d\tau^2}-2\omega\frac{dy_1}{d\tau}&=(\omega^2+2A_1)x_1+2B_1x_3,\\ \frac{d^2x_3}{d\tau^2}-2\omega\frac{dy_3}{d\tau}&=2B_3x_1+(\omega^2+2A_3)x_3,\\ \frac{d^2y_1}{d\tau^2}+2\omega\frac{dx_1}{d\tau}&=(\omega^2-A_1)y_1-B_1y_3,\\ \frac{d^2y_3}{d\tau^2}+2\omega\frac{dx_3}{d\tau}&=-B_3y_1+(\omega^2-A_3)y_3,\end{aligned}\right\}\tag{6}$$

$$\left.\begin{aligned}\frac{d^2z_1}{d\tau^2} &= -A_1z_1 - B_1z_3,\\ \frac{d^2z_3}{d\tau^2} &= -B_3z_1 - A_3z_3,\end{aligned}\right\} \tag{7}$$

where

$$A_1 = \frac{m_1+m_2}{r_{12}^3} + \frac{m_3}{r_{13}^3}, \qquad A_3 = \frac{m_1}{r_{13}^3} + \frac{m_2+m_3}{r_{23}^3},$$

$$B_1 = m_3\left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3}\right), \qquad B_3 = m_1\left(\frac{1}{r_{23}^3} - \frac{1}{r_{13}^3}\right),$$

and the r_{ij} belong to the circular solutions.

Equations (6) and (7) are mutually independent; therefore we can treat them separately. Let us substitute

$$x_i = K_ie^{\lambda\tau}, \qquad y_i = L_ie^{\lambda\tau}, \qquad z_i = M_ie^{\sigma\tau}.$$

From equations (6) there results a set of equations linear and homogeneous in K_i and L_i, and from equations (7) a set linear and homogeneous in M_i. In order that there shall exist a solution other than that given by $K_i = L_i = M_i = 0$ we must have

$$\begin{vmatrix} \lambda^2-\omega^2-2A_1, & -2B_1, & -2\lambda\omega, & 0\\ -2B_3, & \lambda^2-\omega^2-2A_3, & 0, & -2\lambda\omega\\ 2\lambda\omega, & 0, & \lambda^2-\omega^2+A_1, & B_1\\ 0, & 2\lambda\omega, & B_3, & \lambda^2-\omega^2+A_3 \end{vmatrix} = 0, \tag{8}$$

and

$$\begin{vmatrix} \sigma^2+A_1, & B_1\\ B_3, & \sigma^2+A_3 \end{vmatrix} = 0. \tag{9}$$

The left member of equation (8) is a function of λ^2 for if λ be changed into $-\lambda$ the function is unchanged. Lagrange* has proved that there are elliptical orbits in which the bodies all remain on a straight line. These ought to appear here as oscillations near the circular solutions. Therefore we expect equations (8) to have a solution $\lambda^2 = -\omega^2$. On substituting $-\omega^2$ for λ^2 in (8) we obtain from the left member

$$\begin{vmatrix} A_1, & B_1\\ B_3, & A_3 \end{vmatrix} \times \begin{vmatrix} \omega^2-A_1, & B_1\\ B_3, & \omega^2-A_3 \end{vmatrix}. \tag{10}$$

The quantities ω^2, A_1, A_3, B_1 and B_3 are not independent, but are related

* Tisserand, "Méchanique Céleste," Vol. 1, Chapter 8; Moulton's "Celestial Mechanics," p. 217.

by the equations which determine the circular solution, namely,

$$\omega^2\xi_1^{(0)} - \frac{\xi_1^{(0)}(m_1 + m_2) + m_3\xi_3^{(0)}}{r_{12}^3} - \frac{m_3(\xi_1^{(0)} - \xi_3^{(0)})}{r_{13}^3} = 0,$$

$$\omega^2\xi_3^{(0)} - \frac{m_1(\xi_3^{(0)} - \xi_1^{(0)})}{r_{13}^3} - \frac{\xi_3^{(0)}(m_2 + m_3) + m_1\xi_1^{(0)}}{r_{23}^3} = 0,$$

which become when expressed in terms of A's and B's

$$\omega^2 - A_1 = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} B_1, \qquad \omega^2 - A_3 = \frac{\xi_1^{(0)}}{\xi_3^{(0)}} B_3. \tag{11}$$

The first part of (10) is clearly positive. If equations (11) are used the second factor vanishes. Therefore $-\omega^2$ is a root of (8). Further, if we put $\lambda^2 = 0$ in (8) the determinant breaks up into the product of two determinants one of which is the second factor of (10). Therefore (8) has a root $\lambda^2 = 0$ and can be reduced to a quadratic in λ^2 the exact form of which is

$$\lambda^4 + \lambda^2(3\omega^2 - A_1 - A_3) + 3\omega^4 + A_1A_3 - 2A_1^2 - 2A_3^2 - 5B_1B_3 = 0. \tag{12}$$

Since B_1 and B_3 are positive and $\xi_1^{(0)}$ has the opposite sign to $\xi_3^{(0)}$ equations (11) show that

$$\omega^2 - A_1 < 0, \qquad \omega^2 - A_3 < 0. \tag{13}$$

Equations (11) also give

$$(\omega^2 - A_1)(\omega^2 - A_3) = B_1B_3. \tag{14}$$

Using the relation (14) the constant term of (12) can be grouped

$$(\omega^2 + A_3)(\omega^2 - A_1) + (\omega^2 + A_1)(\omega^2 - A_3) + A_1(\omega^2 - A_1) \\ + A_3(\omega^2 - A_3) - 4B_1B_3.$$

The constant term of (12) is therefore negative and the solutions for λ^2 are real, one positive and one negative. Call these roots $-\rho_1^2$ and ρ_2^2.

The forms of equations (9) and (10) show that (9) also has the root $\sigma^2 = -\omega^2$. The other root of (9) is $\sigma^2 = \omega^2 - A_1 - A_3$ which is negative. We denote $\omega^2 - A_1 - A_3$ by $-\nu^2$. From these specific values of the roots of (9) it is clear that $-\nu^2$ is less than $-\omega^2$. If λ^2 in (12) is replaced by $-\nu^2$, the left member reduces to $\omega^2(2\omega^2 - A_1 - A_3)$ which is negative. Hence $-\rho_1^2 < -\nu^2$ and the relative magnitude of the roots of (8) and (9) whatever the values of m_1, m_2 and m_3 are as follows:

$$-\rho_1^2 < -\nu^2 < -\omega^2 < 0 < \rho_2^2.$$

The Generating Solution.—Having found the characteristic exponents,

we can write the general solutions of equations (6) and (7). They are

$$\left.\begin{aligned}
x_1&=K_{11}e^{\rho_1 i\tau}+K_{12}e^{-\rho_1 i\tau}+K_{13}e^{\omega i\tau}+K_{14}e^{-\omega i\tau}+K_{15}e^{\rho_2\tau}+K_{16}e^{-\rho_2\tau}+K_{17}+K_{18}\tau,\\
x_3&=K_{21}e^{\rho_1 i\tau}+K_{22}e^{-\rho_1 i\tau}+K_{23}e^{\omega i\tau}+K_{24}e^{-\omega i\tau}+K_{25}e^{\rho_2\tau}+K_{26}e^{-\rho_2\tau}+K_{27}+K_{28}\tau,\\
y_1&=L_{11}e^{\rho_1 i\tau}+L_{12}e^{-\rho_1 i\tau}+L_{13}e^{\omega i\tau}+L_{14}e^{-\omega i\tau}+L_{15}e^{\rho_2\tau}+L_{16}e^{-\rho_2\tau}+L_{17}+L_{18}\tau,\\
y_3&=L_{21}e^{\rho_1 i\tau}+L_{22}e^{-\rho_1 i\tau}+L_{23}e^{\omega i\tau}+L_{24}e^{-\omega i\tau}+L_{25}e^{\rho_2\tau}+L_{26}e^{-\rho_2\tau}+L_{27}+L_{28}\tau,\\
z_1&=M_{11}e^{\omega i\tau}+M_{12}e^{-\omega i\tau}+M_{13}e^{\kappa i\tau}+M_{14}e^{-\kappa i\tau},\\
z_3&=M_{21}e^{\omega i\tau}+M_{22}e^{-\omega i\tau}+M_{23}e^{\kappa i\tau}+M_{24}e^{-\kappa i\tau}.
\end{aligned}\right\}\quad(15)$$

The solutions for x_i and y_i can contain only eight arbitrary constants. We may choose for six of them $L_{11}\cdots L_{16}$. Then the L_{3j}, K_{1j} and K_{3j}, $j=1\cdots 6$, are uniquely determined in terms of $L_{11}\cdots L_{16}$ by any three of the equations

$$\left.\begin{aligned}
(\lambda^2-\omega^2-2A_1)K_{1j}-2B_1K_{3j}-2\lambda\omega L_{1j}&=0,\\
-2B_3K_{1j}+(\lambda^2-\omega^2-2A_3)K_{3j}-2\lambda\omega L_{3j}&=0,\\
2\lambda\omega K_{1j}+(\lambda^2-\omega^2+A_1)L_{1j}+B_1L_{3j}&=0,\\
2\lambda\omega K_{3j}+B_3L_{1j}+(\lambda^2-\omega^2+A_3)L_{3j}&=0\quad(j=1\cdots 6),
\end{aligned}\right\}\quad(16)$$

because $\rho_1 i$, $-\rho_1 i$, ωi, $-\omega i$, ρ_2, and $-\rho_2$ are simple roots of (8) and therefore not all the first minors obtained by suppressing the column belonging to the L_{ij} can vanish.

To find K_{i7}, K_{i8}, L_{i7} and L_{i8} substitute $x_i=K_{i7}+K_{i8}\tau$, $y_i=L_{i7}+L_{i8}\tau$ in the differential equations (6). There results

$$\begin{aligned}
-2\omega L_{18}&=(\omega^2+2A_1)(K_{17}+K_{18}\tau)+2B_1(K_{37}+K_{38}\tau),\\
-2\omega L_{38}&=2B_3(K_{17}+K_{18}\tau)+(\omega^2+2A_3)(K_{37}+K_{38}\tau),\\
2\omega K_{18}&=(\omega^2-A_1)(L_{17}+L_{18}\tau)-B_1(L_{37}+L_{38}\tau),\\
2\omega K_{38}&=-B_3(L_{17}+L_{18}\tau)+(\omega^2-A_3)(L_{37}+L_{38}\tau).
\end{aligned}$$

Since these equations are identities in τ we have

$$\left.\begin{aligned}
2\omega L_{18}+(\omega^2+2A_1)K_{17}+2B_1K_{37}&=0,\\
2\omega L_{38}+2B_3K_{17}+(\omega^2+2A_3)K_{37}&=0,
\end{aligned}\right\}\quad(17)$$

$$\left.\begin{aligned}
2\omega K_{18}-(\omega^2-A_1)L_{17}+B_1L_{37}&=0,\\
2\omega K_{38}+B_3L_{17}-(\omega^2-A_3)L_{37}&=0,
\end{aligned}\right\}\quad(18)$$

$$\left.\begin{aligned}
(\omega^2+2A_1)K_{18}+2B_1K_{38}&=0,\\
2B_3K_{18}+(\omega^2+2A_3)K_{38}&=0,
\end{aligned}\right\}\quad(19)$$

$$\left.\begin{aligned}
(\omega^2-A_1)L_{18}-B_1L_{38}&=0,\\
-B_3L_{18}+(\omega^2-A_3)L_{38}&=0.
\end{aligned}\right\}\quad(20)$$

In (19) the determinant of the K_{i8} is

$$\begin{vmatrix} \omega^2 + 2A_1, & 2B_1 \\ 2B_3, & \omega^2 + 2A_3 \end{vmatrix} \equiv \omega^4 + 2\omega^2(A_1 + A_3) + 4(A_1A_3 - B_1B_3),$$

which is positive. Therefore $K_{18} = K_{38} = 0$. In (20) the determinant of L_{i8} vanishes, therefore

$$L_{38} = \frac{\omega^2 - A_1}{B_1} L_{18} = \frac{B_3}{\omega^2 - A_3} L_{18} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} L_{18}.$$

Since $K_{18} = K_{38} = 0$ it follows from (18) that

$$L_{37} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} L_{17}.$$

Then (17) combined with (14) gives

$$K_{17} = -\frac{2}{3\omega} L_{18}, \qquad K_{37} = -\frac{2}{3\omega}\frac{\xi_3^{(0)}}{\xi_1^{(0)}} L_{18}.$$

Hence we choose L_{17} and L_{18} arbitrarily. In the same way we choose M_{1j} $(j = 1 \cdots 4)$ arbitrarily and determine M_{3j} in terms of M_{1j} by either of the equations

$$(\sigma^2 + A_1)M_{1j} + B_1M_{3j} = 0, \qquad B_3M_{1j} + (\sigma^2 + A_3)M_{3j} = 0,$$

which give

$$M_{31} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} M_{11}, \qquad M_{32} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}} M_{12}, \qquad M_{33} = \frac{\xi_1^{(0)}}{\xi_3^{(0)}}\frac{B_3}{B_1} M_{13},$$
$$M_{34} = \frac{\xi_1^{(0)}}{\xi_3^{(0)}}\frac{B_3}{B_1} M_{14}. \tag{21}$$

When $dx_i/d\tau = y_i = z_i = 0$ $(i = 1, 3)$ at $\tau = 0$, we get for the solutions

$$x_i = K_{i1}\cos\rho_1\tau + K_{i3}\cos\omega\tau + K_{i5}\cosh\rho_2\tau + K_{i7},$$
$$y_i = L_{i1}\sin\rho_1\tau + L_{i3}\sin\omega\tau + L_{i5}\sinh\rho_2\tau + L_{i8}\tau,$$
$$z_i = M_{i1}\sin\omega\tau + M_{i3}\sin\nu\tau.$$

The generating solutions must be periodic. Therefore we have the following cases to consider:

(*a*) ρ_1, ω and ν incommensurable.

Case 1. $x_i = y_i = 0, \quad z_i = M_{i3}\sin\nu\tau.$

Case 2. $x_i = K_{i3}\cos\omega\tau, \quad y_i = L_{i3}\sin\omega\tau, \quad z_i = M_{i1}\sin\omega\tau.$

Case 3. $x_i = K_{i1}\cos\rho_1\tau, \quad y_i = L_{i1}\sin\rho_1\tau, \quad z_i = 0.$

(*b*) ω and ν commensurable but ρ_1 incommensurable with them.

Case 4. $x_i = K_{i3} \cos \omega\tau$, $y_i = L_{i3} \sin \omega\tau$, $z_i = M_{i1} \sin \omega\tau + M_{i3} \sin \nu\tau$.

(*c*) ρ_1 and ν commensurable, but ω incommensurable with them.

Case 5. $x_i = K_{i1} \cos \rho_1\tau$, $y_i = L_{i1} \sin \rho_1\tau$, $z_i = M_{i3} \sin \nu\tau$.

(*d*) ρ_1 and ω commensurable, but ν incommensurable with them.

Case 6. $x_i = K_{i1} \cos \rho_1\tau + K_{i3} \cos \omega\tau$, $y_i = L_{i1} \sin \rho_1\tau + L_{i3} \sin \omega\tau$,
$z_i = M_{i1} \sin \omega\tau$.

(*e*) ρ_1, ω and ν commensurable.

Case 7. $x_i = K_{i1} \cos \rho_1\tau + K_{i3} \cos \omega\tau$, $y_i = L_{i1} \sin \rho_1\tau + L_{i3} \sin \omega\tau$,
$z_i = M_{i1} \sin \omega\tau + M_{i3} \sin \nu\tau$.

Transformation to the Normal Form.—For the existence proofs and for certain parts of the constructions of the solutions the variables of equations (4) are inconvenient. To introduce variables better adapted to our purpose we make a transformation of the form

$$x_1 = \sum_{j=1}^{8} a_{1j}u_j, \qquad x_1' = \sum_{j=1}^{8} a_{2j}u_j, \qquad x_3 = \sum_{j=1}^{8} a_{3j}u_j, \qquad \text{etc.},$$

where the primes denote derivatives with respect to τ. The determinant of the substitution must not be zero. It follows from the general theory of differential equations in which the characteristic equation has a double root zero that it is possible to determine the a_{ij} so that equations (4) assume the form

$$\left.\begin{aligned}
u_1' &= +\,(1+\delta)\rho_1\iota u_1 + (1+\delta)\epsilon P_1(u_j, z_i),\\
u_2' &= -\,(1+\delta)\rho_1\iota u_2 + (1+\delta)\epsilon P_2(u_j, z_i),\\
u_3' &= +\,(1+\delta)\omega\iota u_3 + (1+\delta)\epsilon P_3(u_j, z_i),\\
u_4' &= -\,(1+\delta)\omega\iota u_4 + (1+\delta)\epsilon P_4(u_j, z_i),\\
u_5' &= +\,(1+\delta)\rho_2 u_5 + (1+\delta)\epsilon P_5(u_j, z_i),\\
u_6' &= -\,(1+\delta)\rho_2 u_6 + (1+\delta)\epsilon P_6(u_j, z_i),\\
u_7' &= +\,(1+\delta)u_8 + (1+\delta)\epsilon P_7(u_j, z_i),\\
u_8' &= \qquad\qquad\quad + (1+\delta)\epsilon P_8(u_j, z_i) \quad (j = 1 \cdots 8),\ (i = 1, 3),\\
z_i'' &= (1+\delta)^2\{z_{1i} + \epsilon z_{2i} + \cdots\} \quad (i = 1, 3),
\end{aligned}\right\} \tag{22}$$

where the P_j are power series in the u_j beginning with terms of the second degree. To determine the constants a_{ij} integrate equations (22) after all terms have been dropped from the right members except the linear ones.

The result is

$$u_1 = L_{11}e^{(1+\delta)\rho_1\iota\tau}, \qquad u_2 = L_{12}e^{-(1+\delta)\rho_1\iota\tau}, \qquad u_3 = L_{13}e^{(1+\delta)\omega\iota\tau},$$

$$u_4 = L_{14}e^{-(1+\delta)\omega\iota\tau}, \qquad u_5 = L_{15}e^{(1+\delta)\rho_2\tau}, \qquad u_6 = L_{16}e^{-(1+\delta)\rho_2\tau},$$

$$u_7 = L_{17} + L_{18}(1+\delta)\tau, \qquad u_8 = L_{18}.$$

On adding these eight equations we get the value of y_1 obtained from integrating the linear terms of (4). The values of x_1, x_1', y_1', y_3 and y_3' are found by multiplying the L_{1j} by certain constants obtained from solving equations (16) for K_{ij} and L_{3j} in terms of L_{1j}. The L_{1j} are arbitrary constants of integration; therefore we choose them all equal to unity since this choice obviously does not make the determinant of the substitution zero. This gives the following form for the substitution:

$$\left.\begin{aligned}
x_1 &= a_1(u_1 - u_2) + a_3(u_3 - u_4) + a_5(u_5 - u_6) + a_7u_8,\\
x_1' &= \rho_1\iota a_1(u_1 + u_2) + \omega\iota a_3(u_3 + u_4) + \rho_2 a_5(u_5 + u_6),\\
x_3 &= b_1(u_1 - u_2) + b_3(u_3 - u_4) + b_5(u_5 - u_6) + b_7u_8,\\
x_3' &= \rho_1\iota b_1(u_1 + u_2) + \omega\iota b_3(u_3 + u_4) + \rho_2 b_5(u_5 + u_6),\\
y_1 &= u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8,\\
y_1' &= \rho_1\iota(u_1 - u_2) + \omega\iota(u_3 - u_4) + \rho_2(u_5 - u_6) + u_8,\\
y_3 &= c_1(u_1 + u_2) + c_3(u_3 + u_4) + c_5(u_5 + u_6) + c_7(u_7 + u_8),\\
y_3' &= \rho_1\iota c_1(u_1 - u_2) + \omega\iota c_3(u_3 - u_4) + \rho_2 c_5(u_5 - u_6) + c_7u_8,
\end{aligned}\right\} \tag{23}$$

where the a_j, b_j and c_j $(j = 1, 3, 5)$ are the ratios of the K_{1j}, K_{3j} and L_{3j} to L_{1j}, a_7 and b_7 are the ratios of K_{17} and K_{37} to L_{18}, and c_7 is the ratio of L_{37} to L_{17}.

Equations (22) are the normal forms for the equations of motion. In the normal variables the generating solutions become

(1) $u_j = 0 \quad (j = 1 \cdots 8), \qquad z_i = M_i \sin\nu\tau \quad (i = 1, 3)$,

(2) $u_j = 0 \quad (j = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$
$z_i = M_i \sin\omega\tau \quad (i = 1, 3)$,

(3) $u_1 = ae^{\rho_1\iota\tau}, \qquad u_2 = ae^{-\rho_1\iota\tau}, \qquad u_j = 0 \quad (j = 3, 4, 5, 6, 7, 8),$
$z_i = 0, \quad (i = 1, 3)$,

(4) $u_j = 0 \quad (j = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$
$z_i = M_i \sin\omega\tau + N_i \sin\nu\tau \quad (i = 1, 3)$,

(5) $u_1 = ae^{\rho_1\iota\tau}, \qquad u_2 = ae^{-\rho_1\iota\tau}, \qquad u_j = 0 \quad (j = 3, 4, 5, 6, 7, 8),$
$z_i = M_i \sin\nu\tau \quad (i = 1, 3)$,

(6) $u_1 = a_1e^{\rho_1\iota\tau}, \qquad u_2 = a_1e^{-\rho_1\iota\tau}, \qquad u_3 = a_2e^{\omega\iota\tau}, \qquad u_4 = a_2e^{-\omega\iota\tau},$
$u_j = 0 \quad (j = 5, 6, 7, 8), \qquad z_i = M_i \sin\nu\tau$,

(7) $u_1 = a_1e^{\rho_1\iota\tau}, \qquad u_2 = a_1e^{-\rho_1\iota\tau}, \qquad u_3 = a_2e^{\omega\iota\tau}, \qquad u_4 = a_2e^{-\omega\iota\tau},$
$u_j = 0 \quad (j = 5, 6, 7, 8), \qquad z_i = M_i \sin\omega\tau + N_i \sin\nu\tau \quad (i = 1, 3)$.

The Periodicity Equations for Solutions with Period $2\pi/\nu$.—According to known theorems* on differential equations it is possible to integrate equations (22) as power series ϵ, δ, α_i, β_i and γ_i, where the α_i are the initial values of the u_i; β_i of z_i and $\nu(M_i + \gamma_i)$ of the z_i'. Instead of integrating as a multiple power series in all these parameters, it is more convenient in the computation to develop the solutions in the form

$$u_i = \sum_{j=0}^{\infty} u_{ij}\epsilon^j, \qquad z_i = \sum_{j=0}^{\infty} z_{ij}\epsilon^j, \tag{24}$$

introducing δ in connection with τ, and the α_i, β_i and γ_i by means of the initial conditions.

The equations determining u_{i0} and z_{i0} are

$$\left.\begin{aligned} u_{10}' &= +\,(1+\delta)\rho_1\iota u_{10}, & u_{50}' &= +\,(1+\delta)\rho_2 u_{50},\\ u_{20}' &= -\,(1+\delta)\rho_1\iota u_{20}, & u_{60}' &= -\,(1+\delta)\rho_2 u_{60},\\ u_{30}' &= +\,(1+\delta)\omega\iota u_{30}, & u_{70}' &= +\,(1+\delta)u_{80},\\ u_{40}' &= -\,(1+\delta)\omega\iota u_{40}, & u_{80}' &= 0,\\ z_{10}'' &= -\,A_1 z_{10} - B_1 z_{30}, & z_{30}'' &= -\,B_3 z_{10} - A_3 z_{30}. \end{aligned}\right\} \tag{25}$$

Integrating and imposing the initial conditions we have

$$\begin{aligned} u_{10} &= \alpha_1 e^{+(1+\delta)\rho_1\iota\tau}, & u_{50} &= \alpha_5 e^{+(1+\delta)\rho_2\tau},\\ u_{20} &= \alpha_2 e^{-(1+\delta)\rho_1\iota\tau}, & u_{60} &= \alpha_6 e^{-(1+\delta)\rho_2\tau},\\ u_{30} &+ \alpha_3 e^{+(1+\delta)\omega\iota\tau}, & u_{70} &= \alpha_7 + \alpha_8(1+\delta)\tau,\\ u_{40} &= \alpha_4 e^{-(1+\delta)\omega\iota\tau}, & u_{80} &= \alpha_8, \end{aligned}$$

$$z_{i0} = M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\nu\iota\tau} + M_{i4}e^{-(1+\delta)\nu\iota\tau},$$

where by equations (21) the M_{3j} are expressed in terms of the M_{1j}. The M_{1j} are expressed in terms of the initial conditions by the equations

$$\beta_1 = M_{11} + M_{12} + M_{13} + M_{14},$$

$$\beta_3 = (M_{11} + M_{12})\frac{\omega^2 - A_1}{B_1} + (M_{13} + M_{14})\frac{\nu^2 - A_1}{B_1},$$

$$\frac{\nu(M_1 + \gamma_1)}{(1+\delta)\iota} = (M_{11} - M_{12})\omega + (M_{13} - M_{14})\nu,$$

$$\frac{\nu(M_3 + \gamma_3)}{(1+\delta)\iota} = (M_{11} - M_{12})\omega\frac{\omega^2 - A_1}{B_1} + (M_{13} - M_{14})\nu\frac{\nu^2 - A_1}{B_1}.$$

The solutions of (22) are

* Moulton's "Periodic Orbits," par. 9.

$$
\left.\begin{aligned}
u_1 &= \alpha_1 e^{(1+\delta)\rho_1 \iota\tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= \alpha_2 e^{-(1+\delta)\rho_1 \iota\tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= \alpha_3 e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= \alpha_4 e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2\tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_6 &= \alpha_6 e^{-(1+\delta)\rho_2\tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_7 &= \alpha_7 + \alpha_8(1+\delta)\tau + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
z_i &= M_{i1} e^{(1+\delta)\omega\iota\tau} + M_{i2} e^{-(1+\delta)\omega\iota\tau} + M_{i3} e^{(1+\delta)\nu\iota\tau} + M_{i4} e^{-(1+\delta)\nu\iota\tau}\\
&\qquad\qquad + \epsilon Q_i(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
z_i' &= (1+\delta)\iota[(M_{i1} e^{+(1+\delta)\omega\iota\tau} - M_{i2} e^{-(1+\delta)\omega\iota\tau})\omega\\
&\qquad + (M_{i3} e^{(1+\delta)\nu\iota\tau} - M_{i4} e^{-(1+\delta)\nu\iota\tau})\nu] + \epsilon Q_i'(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}\right\} \quad (28)
$$

where P_i, Q_i and Q_i' are power series in the indicated arguments.

Sufficient conditions for a periodic solution having the period $2\pi/\nu$ are

$$
\left.\begin{aligned}
&u_i\left(\frac{2\pi}{\nu}\right) - u_i(0) = 0 \quad (i = 1 \cdots 8),\\
&z_i\left(\frac{2\pi}{\nu}\right) - z_i(0) = 0, \qquad z_i'\left(\frac{2\pi}{\nu}\right) - z_i'(0) = 0 \quad (i = 1, 3).
\end{aligned}\right\} \quad (29)
$$

The Existence of Solutions with Period $2\pi/\nu$.—Of the ten known integrals, we have already made use of the six connected with the center of gravity. There are four more to be considered, the energy integral and the three integrals of areas. By means of these four integrals it will be shown that four of the twelve equations of (29) are redundant and can be suppressed.

The integrals are explicitly

$$
\left.\begin{aligned}
F_1 &= \sum_{i=1}^{3} m_i(\bar{x}_i\bar{y}_i' - \bar{y}_i\bar{x}_i') - c_1 = 0,\\
F_2 &= \sum_{i=1}^{3} m_i(\bar{y}_i\bar{z}_i' - \bar{z}_i\bar{y}_i') - c_2 = 0,\\
F_3 &= \sum_{i=1}^{3} m_i(\bar{z}_i\bar{x}_i' - \bar{x}_i\bar{z}_i') - c_3 = 0,\\
F_4 &= \tfrac{1}{2}\sum_{i=1}^{3} [(\bar{x}_i')^2 + (\bar{y}_i')^2 + (\bar{z}_i')^2],
\end{aligned}\right\} \quad (30)
$$

where the $\bar{x}_i$, $\bar{y}_i$ and $\bar{z}_i$ are the rectangular coördinates referred to fixed axes whose origin is at the center of gravity of the system. These variables are related to those used in (22) through the equations

$$
\left.\begin{aligned}
\bar{x}_i &= \xi_i \cos\omega t - \eta_i \sin\omega t,\\
\bar{y}_i &= \xi_i \sin\omega t + \eta_i \cos\omega t,\\
\xi_i &= \xi_i^{(0)} + \epsilon x_i, \qquad \eta_i = \epsilon y_i,\\
\bar{z}_i &= \epsilon z_i, \qquad t = (1+\delta)\tau, \text{ equations (28)}.
\end{aligned}\right\} \quad (31)
$$

Now let $u_i = \alpha_i + v_i$ $(i = 1 \cdots 8)$,

$$z_1 = (M_1 + \gamma_1) \sin \nu\tau + \zeta_1, \qquad z_3 = (M_3 + \gamma_3) \sin \nu\tau + \zeta_3,$$
$$z_1' = \nu(M_1 + \gamma_1) \cos \nu\tau + \zeta_1', \qquad z_3' = \nu(M_3 + \gamma_3) \cos \nu\tau + \zeta_3',$$

where, from (21), $M_3 = [(\nu^2 - A_1)/B_1]M_1$ and where M_1 is taken distinct from zero. It follows from the initial conditions adopted at the beginning of this article that

$$v_1(0) = v_2(0) = \cdots v_8(0) = \zeta_1(0) = \zeta_3(0) = \zeta_1'(0) = \zeta_3'(0) = 0.$$

Since c_1, c_2, c_3 and h can be expressed in terms of the α_i, β_i and γ_i as power series, the integrals can be written in the form

$$F_i = F_i(v_1 \cdots v_8, \zeta_1, \zeta_3, \zeta_1', \zeta_3', \alpha_1 \cdots \alpha_8, \beta_1, \beta_3, \gamma_1, \gamma_3) = 0 \quad (i = 1 \cdots 4),$$

where ϵ is divided out when it enters as a factor. These equations are satisfied at $\tau = 2\pi/\nu$ by $v_1 = \cdots = v_8 = \zeta_1 = \zeta_3 = \zeta_1' = \zeta_3' = 0$ whatever may be the values of $\alpha_1, \cdots \alpha_8, \beta_1, \beta_3, \gamma_1$ and γ_3. For this value of τ they can be solved for v_8, ζ_3, ζ_1', and ζ_3' as power series in $v_1 \cdots v_7, \zeta_1, \alpha_1 \cdots \alpha_8$, $\beta_1, \beta_3, \gamma_1$ and γ_3, and the solutions will vanish for $v_1 = \cdots = v_7 = \zeta_1 = 0$ whatever $\alpha_1 \cdots \alpha_1, \beta_1, \beta_3, \gamma_1$ and γ_3 may be, provided the determinant

$$D = \begin{vmatrix} \frac{\partial F_1}{\partial \nu_8}, & \frac{\partial F_1}{\partial \zeta_3}, & \frac{\partial F_1}{\partial \zeta_1'}, & \frac{\partial F_1}{\partial \zeta_3'} \\ \frac{\partial F_2}{\partial \nu_8}, & \frac{\partial F_2}{\partial \zeta_3}, & \frac{\partial F_2}{\partial \zeta_1'}, & \frac{\partial F_2}{\partial \zeta_3'} \\ \frac{\partial F_3}{\partial \nu_8}, & \frac{\partial F_3}{\partial \zeta_3}, & \frac{\partial F_3}{\partial \zeta_1'}, & \frac{\partial F_3}{\partial \zeta_3'} \\ \frac{\partial F_4}{\partial \nu_8}, & \frac{\partial F_4}{\partial \zeta_3}, & \frac{\partial F_4}{\partial \zeta_1'}, & \frac{\partial F_4}{\partial \zeta_3'} \end{vmatrix} \tag{32}$$

is distinct from zero for $v_1 = \cdots = v_8 = \zeta_1 = \zeta_3 = \zeta_1' = \zeta_3' = \alpha_1 \cdots = \alpha_8 = \beta_1 = \beta_3 = \gamma_1 = \gamma_3 = \delta = \epsilon = 0$. Since F_1 does not involve z_1, z_3 or z_3' all the elements in the first line except the first one are zero. Hence it is only necessary to consider the first diagonal element and its co-factor. In computing the elements of the determinant it is necessary to eliminate the coördinates and derivatives with the subscript two by means of the center of gravity equations. Then it is found from the explicit forms of the integrals and the transformations (23) that

$$\frac{\partial F_1}{\partial \nu_8} = m_1(1 + 2\omega a_7)(\xi_1^{(0)} - \xi_2^{(0)}) + m_3(2\omega b_7 + c_7)(\xi_3^{(0)} - \xi_2^{(0)}),$$

$$\frac{\partial F_2}{\partial \zeta_3} = -\, m_3\omega(\xi_3^{(0)} - \xi_2^{(0)}) \cos\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_2}{\partial \zeta_1'} = m_1(\xi_1^{(0)} - \xi_2^{(0)}) \sin\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_2}{\partial \zeta_3'} = m_3(\xi_3^{(0)} - \xi_2^{(0)}) \sin\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \zeta_3} = -\, m_3\omega(\xi_3^{(0)} - \xi_2^{(0)}) \sin\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \zeta_1'} = -\, m_1(\xi_1^{(0)} - \xi_2^{(0)}) \cos\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \zeta_3'} = -\, m_3(\xi_3^{(0)} - \xi_2^{(0)}) \cos\frac{2\pi\omega}{\nu},$$

$$\frac{\partial F_4}{\partial \zeta_3} = 0,$$

$$\frac{\partial F_4}{\partial \zeta_1'} = \frac{m_1}{m_2}\nu[(m_1 + m_2)M_1 + m_3 M_3],$$

$$\frac{\partial F_4}{\partial \zeta_3'} = \frac{m_3}{m_2}\nu[m_1 M_1 + (m_2 + m_3)M_3],$$

When these elements are substituted in (32) and the determinant is reduced by the elementary rules for the simplification of determinants it is found that

$$D = -\frac{\partial F_1}{\partial \nu_8}\frac{m_1 m_3^2(m_1 + m_2 + m)_3}{m_2}\omega\nu(\xi_3^{(0)} - \xi_2^{(0)})(M_3\xi_1^{(0)} - M_1\xi_3^{(0)}).$$

The factors of this expression are certainly all distinct from zero except the first and last two, which must be considered further. There is no loss of generality in so choosing the notation that $\xi_1^{(0)} < \xi_2^{(0)} < \xi_3^{(0)}$. Then it follows from the center of gravity equation

$$m_1\xi_1^{(0)} + m_2\xi_2^{(0)} + m_3\xi_3^{(0)} = 0$$

that $\xi_1^{(0)}$ is necessarily negative, $\xi_3^{(0)}$ is necessarily positive, and the next to the last factor of D is positive. It was remarked in (13) that $\omega^2 < A_3$. It follows from (9) that $\nu^2 + \omega^2 = A_1 + A_3$. Therefore $\nu^2 - A_1 > 0$ and M_3 has the sign of M_1/B_1. It follows from the definition of B_1 that it is positive. Hence both terms in the last factor of D have the same sign and this factor also is distinct from zero.

It only remains to consider $\partial F_1/\partial \nu_8$. It is found from equations (16)

and (21) that

$$a_7 = \frac{4\omega(\omega^2 - A_1) - 2\omega(\omega^2 + 2A_3)}{(\omega^2 + 2A_1)(\omega^2 + 2A_3) - 4B_1B_3} \equiv -\frac{2}{3\omega},$$

$$b_7 = \frac{-2\omega(\omega^2 + 2A_1)(\omega^2 - A_1) + 4\omega B_1B_3}{B_1[(\omega^2 + 2A_1)(\omega^2 + 2A_3) - 4B_1B_3]} \equiv -\frac{2}{3\omega}\frac{\xi_3^{(0)}}{\xi_1^{(0)}},$$

$$c_7 = \frac{\omega^2 - A_1}{B_1} = \frac{B_3}{\omega^2 - A_3} = \frac{\xi_3^{(0)}}{\xi_1^{(0)}}.$$

When these expressions are simplified it follows that

$$1 + 2\omega a_7 = -\frac{1}{3}, \qquad 2\omega b_7 + c_7 = -\frac{3\xi_3^{(0)}}{\xi_1^{(0)}},$$

and therefore that both terms of $\partial F_1/\partial\nu_8$ are positive. Hence D is distinct from zero.

Since the four integrals can be solved at $t = 2\pi/\nu$ for ν_8, ζ_8, ζ_1' and ζ_3' as power series in $v_1 \cdots v_8$, $\alpha_1 \cdots \alpha_8$, β_1, β_3, γ_1, γ_3 vanishing for $v_1 = \cdots v_7 = \zeta_i = 0$, the conditions

$$u_8\left(\frac{2\pi}{\nu}\right) - u_8 = 0, \quad z_3\left(\frac{2\pi}{\nu}\right) - z_3(0), \quad z_i'\left(\frac{2\pi}{\nu}\right) - z_i'(0) = 0 \quad (i = 1, 3)$$

are a consequence of the remainder of (29), and therefore can be suppressed. The periodicity conditions are then

$$\left.\begin{aligned}
0 &= \alpha_1[e^{(1+\delta)\rho_1 iT} - 1] + \epsilon P_1(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon) \quad \begin{pmatrix} j = 1 \cdots 8 \\ k = 1, 3 \end{pmatrix},\\
0 &= \alpha_2[e^{-(1+\delta)\rho_1 iT} - 1] + \epsilon P_2(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_3[e^{(1+\delta)\omega iT} - 1] + \epsilon P_3(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_4[e^{-(1+\delta)\omega iT} - 1] + \epsilon P_4(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_5[e^{(1+\delta)\rho_2 T} - 1) + \epsilon P_5(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_6[e^{-(1+\delta)\rho_2 T} - 1] + \epsilon P_6(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= \alpha_8 T + \epsilon P_7(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),\\
0 &= z_1(2\pi/\nu) - z_1(0) = M_1 \sin 2\pi(1 + \delta) + Q(\alpha_j, \beta_k, \gamma_k, \delta, \epsilon),
\end{aligned}\right\} \quad (33)$$

where Q vanishes for $\beta_k = \gamma_k = \epsilon = 0$ whatever the α_j and δ may be.

The first seven equations of (33) can be uniquely solved for $\alpha_1 \cdots \alpha_6$ and α_8 as power series in α_7, β_1, β_3, γ_1, γ_3, δ and ϵ vanishing for these quantities equal to zero. Indeed, the solutions vanish for $\epsilon = 0$. Suppose the results are substituted in the last equation of (33); it will then become a

function of α_7, β_1, β_3, γ_1, γ_3, δ and ϵ, vanishing with these quantities. The coefficient of δ to the first power comes from the first term alone and is M_1, which has been taken distinct from zero. Therefore, this equation can be solved for δ as power series in α_7, β_k, γ_k and ϵ, vanishing with these quantities. That is, equations (33) are uniquely solvable for $\alpha_1 \cdots \alpha_6$, α_8 and δ as power series in α_7, β_k, γ_k and ϵ, vanishing with these quantities. Therefore the periodic solutions having the period $2\pi/\nu$ exist.

Orthogonal Orbits with Period $2\pi/\nu$.—We have demonstrated the existence of a unique set of solutions with the period $2\pi/\nu$ by starting from general initial conditions. In the present article we shall show that all orbits of this type cross the x-axis at right angles. The conditions for an orthogonal start are $x_i' = y_i = z_i$ at $t = 0$, or

$$\left.\begin{aligned}
x_1' = 0 &= (1+\delta)\rho_1 \iota a_1(\alpha_1+\alpha_2) + (1+\delta)\omega\iota a_3(\alpha_3+\alpha_4) \\
&\qquad + (1+\delta)\rho_2 a_5(\alpha_5+\alpha_6), \\
x_3' = 0 &= (1+\delta)\rho_1 \iota b_1(\alpha_1+\alpha_2) + (1+\delta)w\iota b_3(\alpha_3+\alpha_4) \\
&\qquad + (1+\delta)\rho_2 b_5(\alpha_5+\alpha_6), \\
y_1 = 0 &= \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \\
y_3 = 0 &= c_1(\alpha_1+\alpha_2)+c_3(\alpha_3+\alpha_4)+c_5(\alpha_5+\alpha_6)+c_7(\alpha_7+\alpha_8), \\
z_1 = 0 &= M_{11}+M_{12}+M_{13}+M_{14}, \\
z_3 = 0 &= M_{31}+M_{32}+M_{33}+M_{34}.
\end{aligned}\right\} \quad (34)$$

The determinant of the $\alpha_i + \alpha_j$ of the right members of the first four equations cannot vanish for otherwise the determinant of equations (23) would vanish. Therefore the only solution of them is

$$\alpha_1+\alpha_2 = 0, \qquad \alpha_3+\alpha_4 = 0, \qquad \alpha_5+\alpha_6 = 0, \qquad \alpha_7+\alpha_8 = 0.$$

Similarly, since M_{3j} are expressible in terms of M_{1j} it follows that

$$M_1 = -M_{12}, \qquad M_{13} = -M_{14}.$$

Put

$$\alpha_1 = -\alpha_2 = \alpha', \quad \alpha_3 = -\alpha_4 = \alpha'', \quad \alpha_5 = -\alpha_6 = \alpha''', \quad \alpha_7 = -\alpha_8 = \alpha^{\text{iv}}.$$

By the symmetry theorem the conditions that the orbits be periodic are

$$x_i' = y_i = z_i \quad \text{at} \quad t = \frac{\pi}{\nu}.$$

These conditions are explicitly

$$\left.\begin{aligned}
0 &= 2(1+\delta)\rho_1\iota a_1\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2(1+\delta)\omega\iota a_3\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) \\
&\quad + 2(1+\delta)\rho_2 a_5\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) + \epsilon R_1(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2(1+\delta)\rho_1\iota b_1\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2(1+\delta)\omega\iota b_3\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) \\
&\quad + 2(1+\delta)\rho_2 b_5\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) + \epsilon R_2(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) + 2\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) \\
&\qquad\qquad + 2\alpha^{\text{iv}} + \epsilon R_3(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2c_1\alpha' \sin \rho_1 \frac{\pi}{\nu}(1+\delta) + 2c_3\alpha'' \sin \omega \frac{\pi}{\nu}(1+\delta) \\
&\quad + 2c_5\alpha''' \sinh \rho_2 \frac{\pi}{\nu}(1+\delta) + 2c_7\alpha^{\text{iv}} + cR_4(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2M_{11} \sin \omega \frac{\pi}{\nu}(1+\delta) + 2M_{13} \sin \pi(1+\delta) \\
&\qquad\qquad + \epsilon R_5(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon), \\
0 &= 2M_{31} \sin \omega \frac{\pi}{\nu}(1+\delta) + 2M_{33} \sin \pi(1+\delta) \\
&\qquad\qquad + \epsilon R_6(\alpha', \alpha'', \alpha''', \alpha^{\text{iv}}, \gamma_i, \delta, \epsilon).
\end{aligned}\right\} \quad (35)$$

There are six equations and eight parameters α', α'', α''', α^{iv}, γ_i, δ, ϵ. Therefore two of them will remain undetermined when (35) are solved. We choose γ and ϵ arbitrarily. The determinant of the linear terms in α', α'', α''', α^{iv} is different from the determinant of equations (34) only by having $16 \sin \rho_1(\pi/\nu)(1+\delta) \sin \omega(\pi/\nu)(1+\delta) \sinh \rho_2(\pi/\nu)(1+\delta)$ as a factor. Since ρ_1, ω, ρ_2 and ν are by hypothesis incommensurable, the determinant does not vanish. We can therefore solve the first four equations uniquely for α', α'', α''', α^{iv} as power series in γ_i, δ, ϵ. Let the results of these solutions be substituted in the last two equations. The determinant of the linear terms in γ_3, δ is

$$\sin \omega \frac{\pi}{\nu} \begin{vmatrix} \dfrac{\nu B_1}{\omega\iota(\omega^2-\nu^2)}, & -\pi M_1 \\ -\dfrac{B_1}{(\omega^2-\nu^2)\iota}, & -\dfrac{\pi M_1(\nu^2 - A_1)}{B_1} \end{vmatrix}.$$

which simplifies to

$$- (\nu^2 - A_1) \frac{(\nu + \omega B_1)\pi M_1 \sin \omega \dfrac{\pi}{\nu}}{\omega(\omega^2 - \nu^2)\iota}.$$

Since ω and ν are incommensurable this cannot vanish. Therefore we can solve the last two equations uniquely for γ_3 and δ as power series in γ_1 and ϵ. For any particular set of values of γ_1 and ϵ there is only one general solution and only one orthogonal solution. Therefore all solutions are orthogonal.

Interpretation of the Arbitrary Constants of the Solution.—The five initial constants α_7, β_1, β_3, γ_1, γ_3 and ϵ were chosen arbitrarily. The ϵ from the way it was introduced determines the magnitude of the deviations from the circular orbits; that is, the relative scale. It is properly called the relative scale, for whenever an orbit is found with a definite value of ϵ there is an infinity of others of different dimensions of the same general shape and the same properties.

The absolute scale of the circular orbits from which the bodies deviate is arbitrary. It is obvious, therefore, that deviations from given circular orbits can be made in such a way that the solutions still remain circular. That is, the final solutions depend on one parameter which is involved in the determination of the absolute scale of the orbits.

If the coordinate axes are not so chosen that the line of the bodies is the x-axis and their plane the xy-plane, then it is always possible by properly determining three constants to rotate to this position. Therefore three of the initial constants account for the position of the coördinate axes.

We are free to choose the origin of time, therefore another of the initial constants is determined by its choice.

To sum up, for general initial conditions there are six arbitrary parameters: two connected with the position of the plane of the bodies, one with the position of their line in this plane, one with the origin of time, and one each with the relative and absolute scales.

If the orbits existed only in the plane of initial motion then the two constants going with the position of this plane would not enter. In the orthogonal existence proofs we have determined all these constants except the relative and absolute scales. A similar interpretation of the arbitraries can be made in all the following cases:

Case 2. *Existence of Orbits with Period* $2\pi/\omega$.—We will now consider the analytic continuation of

$$x_i = K_i \cos \omega\tau, \qquad y_i = L_i \sin \omega\tau, \qquad z_i = M_i \sin \omega\tau,$$

or in the normal variables

$$u_i = 0 \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$$
$$z_i = M_i \sin \omega\tau \quad (i = 1, 3).$$

Let the initial conditions be

$$u_i = \alpha_i \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = a + \alpha_3, \qquad u_4 = a + \alpha_4,$$
$$z_i = \beta_i, \qquad z_i' = c_i + \gamma_i, \qquad c_i = \omega M_i \quad (i = 1, 3).$$

Integrate the equations of motion as power series in ϵ, introducing α_i, β_i, γ_i, δ as in the previous case. The solutions are exactly the same as equations (28) except the u_3 and u_4 equations which are:

$$u_3 = (a + \alpha_3)e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$
$$u_4 = (a + \alpha_4)e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon).$$

Sufficient conditions that the orbits shall be periodic with period $2\pi/\omega$ are

$$u_i\left(\frac{2\pi}{\omega}\right) - u_i(0) = 0 \quad (i = 1 \cdots 8),$$

$$z_i\left(\frac{2\pi}{\omega}\right) - z_i(0) = 0, \qquad z_i'\left(\frac{2\pi}{\omega}\right) - z_i'(0) = 0 \quad (i = 1, 3).$$

Four of these periodicity equations are redundant. The argument is essentially the same as in par. 10. The explicit forms of the integrals are given by equations (30). We propose to show that the equations coming from u_3, u_3, z_3 and z_3' can be suppressed by means of these integrals. Let

$$u_i = \alpha_i + \nu_i \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = (a + \alpha_3)e^{\omega\iota\tau} + \nu_3,$$
$$u_4 = (a + \alpha_4)e^{-\omega\iota\tau} + \nu_4, \qquad z_i = (M_i + \gamma_i) \sin \omega\tau + \zeta_i,$$
$$z_i' = \omega(M_i + \gamma_i) \cos \omega\tau + \zeta_i' \quad (i = 1, 3),$$

where $M_3 = \xi_3^{(0)}/\xi_1^{(0)} M_1$ and $M_1 \neq 0$, $a \neq 0$. Following the argument of par. 10 the equations indicated can be suppressed provided the determinant

$$D = \begin{vmatrix} \dfrac{\partial F_1}{\partial \nu_3}, & \dfrac{\partial F_1}{\partial \nu_3}, & \dfrac{\partial F_1}{\partial \zeta_3}, & \dfrac{\partial F_1}{\partial \zeta_3'} \\ \dfrac{\partial F_2}{\partial \nu_3}, & \dfrac{\partial F_2}{\partial \nu_3}, & \dfrac{\partial F_2}{\partial \zeta_3}, & \dfrac{\partial F_2}{\partial \zeta_3'} \\ \dfrac{\partial F_3}{\partial \nu_3}, & \dfrac{\partial F_3}{\partial \nu_3}, & \dfrac{\partial F_3}{\partial \zeta_3}, & \dfrac{\partial F_3}{\partial \zeta_3'} \\ \dfrac{\partial F_4}{\partial \nu_3}, & \dfrac{\partial F_4}{\partial \nu_3}, & \dfrac{\partial F_4}{\partial \zeta_3}, & \dfrac{\partial F_4}{\partial \zeta_3'} \end{vmatrix} \neq 0.$$

The values of the elements of this determinant computed from equations

(30), (31) and (38) are as follows:

$$\frac{\partial F_1}{\partial \zeta_3}=\frac{\partial F_1}{\partial \zeta_3'}=\frac{\partial F_2}{\partial \zeta_3'}=\frac{\partial F_3}{\partial \zeta_3}=\frac{\partial F_3}{\partial \nu_8}=\frac{\partial F_3}{\partial \nu_3}=\frac{\partial F_4}{\partial \zeta_3'}=0,$$

$$\frac{\partial F_3}{\partial \zeta_3'}=-m_3(\xi_3^{(0)}-\xi_2^{(0)}),\qquad \frac{\partial F_2}{\partial \zeta_3}=-\omega m_3(\xi_3^{(0)}-\xi_2^{(0)}),$$

$$\frac{\partial F_1}{\partial \nu_8}=m_1(\xi_1^{(0)}-\xi_2^{(0)})(1+2\omega a_7)+m_3(\xi_3^{(0)}-\xi_2^{(0)})(c_7+2\omega b_7),$$

$$\frac{\partial F_1}{\partial \nu_3}=m_1(\xi_1^{(0)}-\xi_2^{(0)})(\omega\iota+2\omega a_3)+m_3(\xi_3^{(0)}-\xi_2^{(0)})(\omega\iota c_3+2\omega b_3),$$

$$\frac{\partial F_4}{\partial \nu_8}=2\omega^2 m_1(\xi_1^{(0)}-\xi_2^{(0)})a_7+2\omega^2 m_3(\xi_3^{(0)}-\xi_2^{(0)})b_7,$$

$$\frac{\partial F_4}{\partial \nu_3}=2\omega^2 m_1(\xi_1^{(0)}-\xi_2^{(0)})a_3+2\omega^2 m_3(\xi_3^{(0)}-\xi_2^{(0)})b_3.$$

Therefore D reduces to

$$\frac{\partial F_2}{\partial \zeta_3}\cdot\frac{\partial F_3}{\partial \zeta_3'}\begin{vmatrix}\dfrac{\partial F_1}{\partial \nu_8}, & \dfrac{\partial F_1}{\partial \nu_3}\\[2ex] \dfrac{\partial F_4}{\partial \nu_8}, & \dfrac{\partial F_4}{\partial \nu_3}\end{vmatrix}.$$

Neither of the first two factors vanish. The third factor easily reduces to

$$\begin{vmatrix} m_1(\xi_1^{(0)}-\xi_2^{(0)})+m_3(\xi_3^{(0)}-\xi_2^{(0)})c_7, & m_1(\xi_1^{(0)}-\xi_2^{(0)})\omega\iota+m_3(\xi_3^{(0)}-\xi_2^{(0)})\omega\iota c_3,\\ m_1(\xi_1^{(0)}-\xi_2^{(0)})a_7+m_3(\xi_3^{(0)}-\xi_2^{(0)})b_7, & m_1(\xi_1^{(0)}-\xi_2^{(0)})a_3+m_3(\xi_3^{(0)}-\xi_2^{(0)})b_7.\end{vmatrix} \tag{40}$$

The values of a_7, b_7 and c_7 are given by equations (32). We give below the values a_1, b_1 and c_1, as obtained from equations (16), noting that a_3, b_3 and c_3 can be obtained from a_1, b_1, c_1 by replacing $\rho_1\iota$ by $\omega\iota$

$$\left.\begin{aligned} a_1 &= \frac{-2\rho_1\omega B_1\iota(\rho_1^2-\omega^2-2A_1-2A_3)}{(\rho_1^2+\omega^2+2A_1)(\rho_1^2+\omega^2+2A_3)(-\rho_1^2-\omega^2+A_1) + 4\rho_1^2\omega^2(\rho_1^2+\omega^2+2A_3)+4B_1B_3(\rho_1^2+\omega^2+A_1)},\\ b_1 &= \frac{2\rho_1\omega\iota\{(\rho_1^2+\omega^2+2A_1)(\rho_1^2+\omega^2-A_1)-2B_1B_3-4\rho_1^2\omega^2\}}{d},\\ c_1 &= \frac{-B_1(\rho_1^2+\omega^2+2A_1)(\rho_1^2+\omega^2+2A_3)+8\rho^2\omega^2B_1+4B_1^2B_3}{d},\end{aligned}\right\} \tag{41}$$

where d represents the denominator of a_1. From these values one finds by a short calculation that

$$b_3=\frac{\xi_1^{(0)}}{\xi_3^{(0)}}a_3,\qquad c_3=\frac{\xi_1^{(0)}}{\xi_3^{(0)}},\qquad a_3=-\frac{\iota}{2}\frac{\xi_3^{(0)}}{\xi_1^{(0)}}.$$

while from equations (32)

$$b_7 = \frac{\xi_1^{(0)}}{\xi_3^{(0)}} a_7.$$

Equation (40) then reduces to

$$\left\{ m_1(\xi_1^{(0)} - \xi_2^{(0)}) + m_3(\xi_3^{(0)} - \xi_2^{(0)}) \frac{\xi_1^{(0)}}{\xi_3^{(0)}} \right\} \begin{vmatrix} 1 & \omega\iota \\ a_7 & a_8 \end{vmatrix}.$$

The first factor does not vanish since every term is negative. We have previously found

$$1 + 2\omega a_7 = -1/3 \qquad \text{or} \qquad 2\omega a_7 = -4/3.$$

Therefore the second factor reduces to

$$-\frac{1}{2}\left[\frac{\xi_3^{(0)}}{\xi_1^{(0)}} - \frac{4}{3}\right].$$

It follows that the determinant (39) does not vanish and the u_8, u_8, z_3, z_3' equations can be suppressed.

The necessary and sufficient conditions for a solution of period $2\pi/\omega$ are therefore

$$\left.\begin{aligned}
0 &= \alpha_1(e^{(1+\delta)\rho_1\iota(2\pi/\omega)} - 1) + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= \alpha_2(e^{-(1+\delta)\rho_1\iota(2\pi/\omega)} - 1) + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= (a + \alpha_4)(e^{-(1+\delta)\iota 2\pi} - 1) + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= \alpha_5(e^{(1+\delta)\rho_2(2\pi/\omega)} - 1) + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= \alpha_6(e^{-(1+\delta)\rho_2(2\pi/\omega)} - 1) + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= \alpha_8\frac{2\pi}{\omega} + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= M_{11}(e^{(1+\delta)2\pi\iota} - 1) + M_{12}(e^{-(1+\delta)2\pi\iota} - 1) + M_{13}(e^{(1+\delta)\nu\iota(2\pi/\omega)} - 1) \\
&\qquad + M_{14}(e^{-(1+\delta)\nu\iota(2\pi/\omega)} - 1) + \epsilon Q(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
0 &= \omega M_{11}(e^{(1+\delta)2\pi\iota} - 1) - \omega M_{12}(e^{-(1+\delta)2\pi\iota} - 1) + \nu M_{13}(e^{(1+\delta)\nu\iota(2\pi/\omega)} - 1) \\
&\qquad - \nu M_{14}(e^{-(1+\delta)\nu\iota(2\pi/\omega)} - 1) + \epsilon Q'(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}\right\} \quad (42)$$

where M_{i1}, M_{i2}, M_{i3}, M_{i4} are expressed in terms of β_i and γ_i by the equations

$$\left.\begin{aligned}
\beta_1 &= M_{11} + M_{12} + M_{13} + M_{14}, \\
\beta_3 &= \frac{\omega^2 - A_1}{B_1}(M_{11} + M_{12}) + \frac{\nu^2 - A_1}{B_1}(M_{13} + M_{14}), \\
c_1 + \gamma_1 &= \omega\iota(M_{11} - M_{12}) + \nu\iota(M_{13} - M_{14}), \\
c_3 + \gamma_3 &= \omega\iota\frac{\omega^2 - A_1}{B_1}(M_{11} - M_{12}) + \nu\iota\frac{(\nu^2 - A_1)}{B_1}(M_{13} - M_{14}).
\end{aligned}\right\} \quad (43)$$

The first six of equations (42) can be solved uniquely for $\alpha_1, \alpha_2, \delta, \alpha_5, \alpha_6, \alpha_8$ as power series in $\epsilon, \alpha_3, \alpha_4, \alpha_7, \beta_i, \gamma_i$ since the determinant of the linear terms in these quantities does not vanish. Suppose these solutions are substituted in the last two equations. Then the determinant of the linear terms in β_1, γ_1 reduces to

$$-4\left(\frac{\nu^2 - A_1}{\nu^2 - \omega^2}\right)^2 \left(e^{\frac{\pi\nu}{\omega}} - e^{-\frac{\pi\nu}{\omega}}\right)^2.$$

Since ω and ν are incommensurable by hypothesis, it follows that this expression does not vanish. Therefore the last two equations can be solved uniquely for β_1 and γ_1, as power series in $\epsilon, \alpha_3, \alpha_4, \alpha_7, \beta_3, \gamma_3$. Thus the existence of an unique set of orbits with period $2\pi/\omega$ is proven. It can be shown that there is an unique set of period $2K(\pi/\omega)$ which include the case $K = 1$. It follows that all orbits of this type are reëntrant after one revolution. Further it can be shown, as in case 1, that a unique set of orthogonal orbits exists. It follows that the orthogonal orbits are the only ones.

Case 3. *Existence of Orbits with the Period* $2\pi/\rho_1$.—Let us consider the analytic continuation of the generating solutions

$$x_i = K_i \cos \rho_1 \tau, \qquad y_i = L_i \sin \rho_1 \tau, \qquad z_i = 0;$$

or, in the normal variables, of

$$u_1 = a e^{\rho_1 \iota \tau}, \qquad u_2 = a e^{-\rho_1 \iota \tau}, \qquad u_i = 0 \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = 0 \quad (i = 1, 3).$$

Let the initial conditions be

$$u_1 = a + \alpha_1, \qquad u_2 = a + \alpha_2, \qquad u_i = \alpha_i \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = \beta_i, \qquad z_i' = \gamma_i \quad (i = 1, 3).$$

The solutions of the differential equations with these initial conditions are

$$\begin{aligned}
u_1 &= (a + \alpha_1) e^{(1+\delta)\rho_1 \iota \tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= (a + \alpha_2) e^{-(1+\delta)\rho_1 \iota \tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= \alpha_3 e^{(1+\delta)\omega \iota \tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= \alpha_4 e^{-(1+\delta)\omega \iota \tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2 \tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_6 &= \alpha_6 e^{-(1+\delta)\rho_2 \tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_7 &= \alpha_7 + \alpha_8 \tau + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}$$

$$z_i = M_{i1}e^{(1+\delta)\omega\tau} + M_{i2}e^{-(1+\delta)\omega\tau} + M_{i3}e^{(1+\delta)\nu\tau} + M_{i4}e^{-(1+\delta)\nu\tau} + \epsilon Q_i(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),$$

$$z_i' = \{\omega M_{i1}e^{(1+\delta)\omega\tau} - \omega M_{i2}e^{-(1+\delta)\omega\tau} + \nu M_{i3}e^{(1+\delta)\nu\tau} - \nu M_{i4}e^{-(1+\delta)\nu\tau}\}(1+\delta)\iota + \epsilon Q_i'(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon).$$

The M_{ij} are homogeneous and linear in the β_i, γ_i. It will be proved that the analytic continuation of this solution exists only in the xy-plane.

The periodicity conditions are of the same form as in the other two cases. Without writing them out in full, it is clear that if $\beta_i = \gamma_i = 0$ the orbits are wholly in the xy-plane. On the other hand, if β_i, γ_i are different from zero, we shall show that the solution is impossible. The linear terms of the z_i-equations are homogeneous in β_i, γ_i with a determinant different from zero. We solve three of them for β_1, β_3 and γ_1 and substitute the results in the fourth equation. Then γ_3 is a factor and can be divided out. There is a term left independent of all the other quantities α_i, δ_i, ϵ. Therefore the solution as a power series in the remaining parameters, vanishing with them, is impossible.

The conditions for periodicity become therefore

$$u_i\left(\frac{2\pi}{\rho_1}\right) - u_i(0) = 0 \quad (i = 1, \cdots, 8).$$

Again these conditions are not independent, but now only two of them are redundant. There are only two integrals when the problem is in the plane, the energy integral and one integral of areas. It can be shown as in case (1) that the u_2- and u_7-equations are consequences of the others.

The suppression of these two equations is possible provided the Jacobian

$$D = \begin{vmatrix} \frac{\partial F_1}{\partial \nu_8}, & \frac{\partial F_1}{\partial \nu_2} \\ \frac{\partial F_4}{\partial \nu_8}, & \frac{\partial F_4}{\partial \nu_2} \end{vmatrix}$$

is different from zero for zero values of the initial constants, δ and ϵ. From the explicit values of the integrals (equations 30) we find

$$\frac{\partial F_1}{\partial \nu_8} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(2\omega a_7 + 1) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(2\omega b_7 + c_7),$$

$$\frac{\partial F_1}{\partial \nu_2} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(2\omega a_1 + \rho_1\iota) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(2\omega b_1 + c_1\rho_1\iota),$$

$$\frac{\partial F_4}{\partial \nu_8} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_7 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_7\},$$

$$\frac{\partial F_4}{\partial \nu_2} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_1 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_1\}.$$

The values of $2\omega a_7 + 1$ and $2\omega b_7 + c_7$ are given in equations (32). Multiplying the first row of D by 2, subtracting the second row from it and expanding we obtain

$$D = -\frac{2\omega}{3}\left\{ m_1(\xi_1^{(0)} - \xi_2^{(0)}) + m_3(\xi_3^{(0)} - \xi_2^{(0)})\frac{\xi_3^{(0)}}{\xi_1^{(0)}} \right\}\{m_1(\xi_1^{(0)} - \xi_2^{(0)})(3\omega a_1 + \rho_1\iota) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(3\omega b_1 + 2c_1\rho_1\iota)\}.$$

The first two factors are clearly negative. We shall prove that $3\omega a_1 + \rho_1\iota$ is opposite in sign to $3\omega b_1 + 2c_1\rho_1\iota$. The values of a_1, b_1 and c_1 are

$$a_1 = \frac{2\rho_1\omega\iota\{\rho_1^2 + \omega^2 - 2A_1 - 2A_3\}}{(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 + 2A_3) - 8\rho_1^2\omega^2 - 4B_1B_3} = \frac{E}{d},$$

$$b_1 = -\frac{2\rho_1\omega\iota\{(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 - A_1) - 4\rho_1^2\omega^2 - 2B_1B_3\}}{d} = \frac{F}{d},$$

$$c_1 = \frac{(\rho_1^2 + \omega^2 + 2A_1)(\rho_1^2 + \omega^2 + 2A_3)(\rho_1^2 + \omega^2 - A_1) - 4\rho_1^2\omega^2(\rho_1^2 + \omega^2 + 2A_3) - 4B_1B_3(\rho_1^2 + \omega^2 - A_1)}{d} = \frac{G}{d}.$$

To determine the sign of $3\omega E + 2\rho_1\iota d$ we first divide out the factor $2\rho_1\iota$, then substitute for ρ_1^4 its value in terms of ρ_1^2 given by equation (12). After considerable simplification there results

$$\omega^4 - 5\omega^2(A_1 + A_3) + \rho_1^2(A_1 + A_3) + 4A_1A_3 + 2A_1^2 + 2A_3^2.$$

By substituting $\lambda^2 = -A_1$ and $\lambda^2 = -A_3$ in (12) we find $\rho_1^2 > A_1$ and $\rho_1^2 > A_3$. We have already proven $A_1 > \omega^2$ and $A_3 > \omega^2$. It follows that the last four terms are greater than $5\omega^2(A_1 + A_3)$, therefore $3\omega a_1 + \rho_1\iota$ is of the same sign as d. After eliminating ρ_1^4 by (12) and simplifying we find

$$b_1 = -\frac{2\rho_1\omega\iota(\omega^2 - A_3)(\rho_1^2 + \omega^2 - 2A_1 - 2A_3)}{d},$$

$$c_1 = \frac{3\rho_1^2\omega^2 - (A_1 + A_3)(\rho_1^2 + \omega^2 - 2A_1 - 2A_3)}{d}.$$

Then

$$3\omega b_1 + \rho_1\iota c_1 = \frac{2\rho_1\iota(\omega^2 - A_3)[-3\omega^4 + 2\omega^2(A_1 + A_3) + (A_1 + A_3)(-\rho_1^2 + 3\omega^2 + 2A_1 + 2A_3)]}{d}.$$

It follows that $3\omega b_1 + \rho_1\iota c_1$ has the sign opposite to that of d. Since $\xi_1^{(0)} - \xi_2^{(0)}$ is negative and $\xi_3^{(0)} - \xi_2^{(0)}$ is positive, all the terms in the third factor of D are of the same sign. Therefore the u_2 and u_3 equations can be suppressed.

Necessary and sufficient conditions that the orbits shall be periodic with the period $2\pi/\rho_1$ become

$$\left.\begin{aligned}
0 &= (a+\alpha_1)(e^{(1+\delta)\iota 2\pi} - 1) + \epsilon P_1(\alpha_i, \delta, \epsilon),\\
0 &= \alpha_3(e^{(1+\delta)\omega\iota(2\pi/\rho_1)} - 1) + \epsilon P_3(\alpha_i, \delta, \epsilon),\\
0 &= \alpha_4(e^{-(1+\delta)\omega\iota(2\pi/\rho_1)} - 1) + \epsilon P_4(\alpha_i, \delta, \epsilon),\\
0 &= \alpha_5(e^{(1+\delta)\rho_2(2\pi/\rho_1)} - 1) + \epsilon P_5(\alpha_i, \delta, \epsilon),\\
0 &= \alpha_6(e^{-(1+\delta)\rho_2(2\pi/\rho_1)} - 1) + \epsilon P_6(\alpha_i, \delta, \epsilon),\\
0 &= \alpha_8(2\pi/\rho_1) + \epsilon P_8(\alpha_i, \delta, \epsilon).
\end{aligned}\right\} \tag{45}$$

There are six equations and the ten parameters, $\alpha_1 \cdots \alpha_8, \delta, \epsilon$. Therefore we can choose $\alpha_1, \alpha_2, \alpha_8, \epsilon$ arbitrarily and solve for the others as power series in $\alpha_1, \alpha_2, \alpha_8, \epsilon$ vanishing with these arguments. The equations have unique solutions since the determinant of the linear terms in $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \delta$ is

$$a(e^{(1+\delta)\omega\iota(2\pi/\rho_1)} - 1)(e^{-(1+\delta)\omega\iota(2\pi/\rho_1)} - 1)(e^{(1+\delta)\rho_2(2\pi/\rho_1)} - 1)(e^{-(1+\delta)\rho_2(2\pi/\rho_1)} - 1)\frac{2\pi}{\rho_1} \neq 0$$

and since $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \delta = 0$ is not a solution. This proves the existence of a unique set of orbits in the plane with the period $2\pi/\rho_1$.

We can show, as in cases (1) and (2), that all orbits of this type are reëntrant after one revolution, and are orthogonal.

Cases 4, 6 *and* 7.—The generating solution for these orbits is

$$x_i = K_i \cos \omega\tau, \qquad y_i = L_i \sin \omega\tau, \qquad z_i = M_i \sin \omega\tau + N_i \sin \nu\tau,$$

or, in the normal variables,

$$u_i = 0 \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = ae^{\omega\iota\tau}, \qquad u_4 = ae^{-\omega\iota\tau},$$
$$z_i = M_i \sin \omega\tau + N_i \sin \nu\tau.$$

Let the initial conditions be

$$u_i = \alpha_i \quad (i = 1, 2, 5, 6, 7, 8), \qquad u_3 = a + \alpha_3, \qquad u_4 = a + \alpha_4,$$
$$z_i = \beta_i, \qquad z_i' = c_i + \gamma_i \quad \text{where} \quad c_i = \omega M_i + \nu N_i.$$

With these initial conditions, the solutions of the differential equations (22) are

$$\begin{aligned}
u_1 &= \alpha_1 e^{(1+\delta)\rho_1\iota\tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= \alpha_2 e^{-(1+\delta)\rho_1\iota\tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= (a+\alpha_3)e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= (a+\alpha_4)e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2\tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned}$$

$$
\begin{aligned}
u_6 &= \alpha_6 e^{-(1+\delta)\rho_3\tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
u_7 &= \alpha_8\tau + \alpha_7 + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon), \\
z_i &= M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\nu\iota\tau} + M_{i4}e^{-(1+\delta)\nu\iota\tau} \\
&\qquad + \epsilon Q_i(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon), \\
z_i' &= (1+\delta)\iota\{\omega M_{i1}e^{(1+\delta)\omega\iota\tau} - \omega M_{i2}e^{-(1+\delta)\omega\iota\tau} + \nu M_{i3}e^{(1+\delta)\nu\iota\tau} \\
&\qquad - \nu M_{i4}e^{-(1+\delta)\nu\iota\tau}\} + \epsilon Q_i'(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon).
\end{aligned} \tag{46}
$$

The M_{ij} are the solutions of the equations

$$\beta_1 = M_{11} + M_{12} + M_{13} + M_{14},$$

$$\beta_3 = \frac{\omega^2 - A_1}{B_1}(M_{11} + M_{12}) + \frac{\nu^2 - A_1}{B_1}(M_{13} + M_{14}),$$

$$\frac{c_1 + \gamma_1}{(1+\delta)\iota} = \omega(M_{11} - M_{12}) + \nu(M_{13} - M_{14}),$$

$$\frac{c_3 + \gamma_3}{(1+\delta)\iota} = \omega\frac{\omega^2 - A_1}{B_1}(M_{11} - M_{12}) + \nu\frac{\nu^2 - A_1}{B_1}(M_{13} - M_{14}),$$

where $c_1 = \omega M_1 + \nu N_1$, $c_3 = \omega M_3 + \nu N_3$. By equations (13) we have the relations

$$M_3 = \frac{\omega^2 - A_1}{B_1}M_1, \qquad N_3 = \frac{\nu^2 - A_1}{B_1}N_1.$$

Making use of these relations, it is found that the M_{ij} are non-homogeneous in β_i and γ_i.

The periodicity conditions

$$u_i(T) - u_i(0) = 0 \quad (i = 1 \cdots 8), \qquad T = \frac{2k\pi}{\omega} = \frac{2k'\pi}{\nu},$$

$$z_i(T) - z_i(0) = 0, \qquad z_i'(T) - z_i'(0) = 0 \quad (i = 1, 3),$$

are not independent, but we do not need to discuss their independence here. We can show that the orbits do not exist unless there are certain relations among the constants of the generating solution. Suppose we solve some of the u_i-equations for α_i, all the remaining α_i to be chosen arbitrarily. We substitute these solutions in the equation coming from u_3. In place of the arbitrary α_j put $\eta_j\epsilon$. It follows that the value of δ coming from the u_3-equation begins with terms in ϵ^2. We substitute this value in the equa-

tions

$$z_i(T) - z_i(0) = 0 \quad (i = 1, 3)$$

and divide out the factor ϵ from each equation. Then, in each equation there is left a term independent of all the initial constants. One might make these terms vanish by choosing the constants of the generating solution properly, but the computation has proved so complicated that a full discussion has not been made.

The same condition of affairs turns up in cases 6 and 7 and we are at once able to say that no orbits exist in these cases except, possibly, when there are special relations among the constants of the generating solution.

Case 5. *Orbits in which* ρ_1 *and* ν *are commensurable.*—We shall prove the existence of orbits of which the period T is the least common integral multiple of $2\pi/\rho_1$ and $2\pi/\nu$. The generating solution is

$$u_1 = ae^{\rho_1 \iota\tau}, \qquad u_2 = ae^{-\rho_1 \iota\tau}, \qquad u_i = 0 \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = M_i \sin \nu\tau \quad (i = 1, 3).$$

Let the initial conditions be

$$u_1 = a + \alpha_1, \qquad u_2 = a + \alpha_2, \qquad u_i = \alpha_i \quad (i = 3, 4, 5, 6, 7, 8),$$
$$z_i = \beta_i, \qquad z_i' = c_i + \gamma_i, \qquad c_i = \nu M_i \quad (i = 1, 3).$$

The solutions of the differential equations (22) with these initial conditions are

$$\begin{aligned}
u_1 &= (a + \alpha_1)e^{(1+\delta)\rho_1\iota\tau} + \epsilon P_1(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_2 &= (a + \alpha_2)e^{-(1+\delta)\iota\tau} + \epsilon P_2(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_3 &= \alpha_3 e^{(1+\delta)\omega\iota\tau} + \epsilon P_3(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_4 &= \alpha_4 e^{-(1+\delta)\omega\iota\tau} + \epsilon P_4(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_5 &= \alpha_5 e^{(1+\delta)\rho_2\tau} + \epsilon P_5(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_6 &= \alpha_6 e^{-(1+\delta)\rho_2\tau} + \epsilon P_6(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_7 &= \alpha_8\tau + \alpha_7 + \epsilon P_7(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),\\
u_8 &= \alpha_8 + \epsilon P_8(\alpha_i, \beta_i, \gamma_i, \delta, \epsilon),
\end{aligned} \tag{47}$$

$$z_i = M_{i1}e^{(1+\delta)\omega\iota\tau} + M_{i2}e^{-(1+\delta)\omega\iota\tau} + M_{i3}e^{(1+\delta)\kappa\tau} + M_{i4}e^{-(1+\delta)\kappa\tau} + \epsilon Q_i(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon),$$

$$z_i' = (1 + \delta)\,\iota\{\omega M_{i1}e^{(1+\delta)\omega\iota\tau} - \omega M_{i2}e^{-(1+\delta)\omega\iota\tau} + \nu M_{i3}e^{(1+\delta)\kappa\tau} - \nu M_{i4}e^{-(1+\delta)\iota\tau}\} + \epsilon Q_i'(\alpha_j, \beta_j, \gamma_j, \delta, \epsilon),$$

where the M_{ij} have the same form as the expressions (25); that is, M_{11}, M_{12} are homogeneous in β_i, γ_i but M_{13}, M_{14} each have a term independent of β_i and γ_i. The periodicity conditions are of the same form as before.

The argument and method of proof is almost identical with case 2. The equations coming from u_8, u_2, z_3 and z_3' can be suppressed provided the Jacobian

$$D = \begin{vmatrix} \frac{\partial F_1}{\partial \nu_8}, & \frac{\partial F_1}{\partial \nu_2}, & \frac{\partial F_1}{\partial \xi_3}, & \frac{\partial F_1}{\partial \xi_3'} \\ \frac{\partial F_2}{\partial \nu_8}, & \frac{\partial F_2}{\partial \nu_2}, & \frac{\partial F_2}{\partial \xi_3}, & \frac{\partial F_2}{\partial \xi_3'} \\ \frac{\partial F_3}{\partial \nu_8}, & \frac{\partial F_3}{\partial \nu_2}, & \frac{\partial F_3}{\partial \xi_3}, & \frac{\partial F_3}{\partial \xi_3'} \\ \frac{\partial F_4}{\partial \nu_8}, & \frac{\partial F_4}{\partial \nu_2}, & \frac{\partial F_4}{\partial \xi_3}, & \frac{\partial F_4}{\partial \xi_3'} \end{vmatrix}$$

is distinct from zero.

The values of the elements of D which are needed are:

$$\frac{\partial F_1}{\partial \xi_3} = \frac{\partial F_1}{\partial \xi_3'} = \frac{\partial F_2}{\partial \nu_8} = \frac{\partial F_2}{\partial \xi_3'} = \frac{\partial F_3}{\partial \xi_3} = \frac{\partial F_4}{\partial \xi_3} = \frac{\partial F_4}{\partial \xi_3'} = 0,$$

$$\frac{\partial F_2}{\partial \xi_3} = -\omega m_3(\xi_3^{(0)} - \xi_2^{(0)}) \cos \frac{2k\pi\omega}{\nu},$$

$$\frac{\partial F_3}{\partial \xi_3'} = -m_3(\xi_3^{(0)} - \xi_2^{(0)}) \cos \frac{2k\pi\omega}{\nu},$$

$$\frac{\partial F_1}{\partial \nu_8} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(1 + 2\omega a_7) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(c_7 + 2\omega b_7),$$

$$\frac{\partial F_1}{\partial \nu_2} = m_1(\xi_1^{(0)} - \xi_2^{(0)})(\rho_1 \iota + 2\omega a_1) + m_3(\xi_3^{(0)} - \xi_2^{(0)})(\rho_1 \iota c_1 + 2\omega b_1),$$

$$\frac{\partial F_4}{\partial \nu_8} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_7 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_7\},$$

$$\frac{\partial F_4}{\partial \nu_2} = \omega^2\{m_1(\xi_1^{(0)} - \xi_2^{(0)})a_1 + m_3(\xi_3^{(0)} - \xi_2^{(0)})b_1\}.$$

Therefore D reduces to

$$-\frac{\partial F_2}{\partial \xi_3} \cdot \frac{\partial F_3}{\partial \xi_3'} \begin{vmatrix} \frac{\partial F_1}{\partial \nu_8}, & \frac{\partial F_1}{\partial \nu_2} \\ \frac{\partial F_4}{\partial \nu_8}, & \frac{\partial F_4}{\partial \nu_2} \end{vmatrix}.$$

Neither of the first two factors vanish and the third factor has already been discussed in case 3. The remainder of the existence proof is identical with case 2.

Comparison of the Commensurable Cases with the Incommensurable.—In case 5 it has been found that the orbits exist for general values of the constants of the generating solution. Necessarily they still exist for particular values. Then in case 5 put $M_1 = 0$. It follows that $M_3 = 0$ and the generating solution becomes the same as in case 3. The conclusion is that the orbits of case 3 exist even when ρ_1 and ν are commensurable. If we put $a = 0$ in case 5, the generating solution becomes the same as in case 1. Hence the orbits of case 1 exist when ρ_1 and ν are commensurable. Cases 4, 6 and 7 have not been completely discussed and consequently the existence of the orbits in cases 1, 2 and 3, when ω and ν are commensurable, when ρ_1 and ω are commensurable, and when ρ_1, ω and ν are commensurable, is not proven.

ON CERTAIN CHAINS OF THEOREMS IN REFLEXIVE GEOMETRY.

By Flora D. Sutton.

§ 1. Introduction.

It is of interest to extend properties of the triangle to more than three lines and, when possible, to n lines of a plane. This can often be done by a process called geometrical interpolation. As an illustration, I consider, in what follows, a problem proposed by Desboves in his "Questions de Géométrie Élementaire," * namely: "Des trois sommets a, b, c d'un triangle on abaisse des perpendiculaires ap, bq, cr sur une droite quelconque de son plan, pius, des points p, q, r des perpendiculaires sur bc, ac, ab: ces trois dernières droites se coupent en un même point." A discussion of this problem, by means of trilinear coördinates, is given by Kazimierz Cwojdzinski in *Archiv der Mathematik und Physik*.†

In the present paper some extensions of this theorem are considered as indicated in the following scheme:

Undirected Lines.

First Chain.		Second Chain.	
1—a.	3 lines	2—a.	4 lines
1—b.	4 lines	2—b.	5 lines
1—c.	5 lines	2—c.	6 lines
1—g.	general statement	2—g.	general statement

Directed Lines.

I—a.	4 lines	II—a.	5 lines
I—b.	5 lines	II—b.	6 lines
I—g.	general statement	II—g.	general statement

We name a point of the plane by a complex number.

§ 2. Some Fundamental Formulæ in Reflexive Geometry.

1. *Reflexion in a Line.*

If the two triangles x, a, b and y, b, a are inversely similar, and y is the reflexion of x, b the reflexion of a, and a the reflexion of b, in one and the same line, then the two triangles x, a, b and $\bar{y}$, $\bar{b}$, $\bar{a}$ are directly similar.

* 1875; page 241; No. 77.

† 1901; Vol. 1; pp. 175–180.

The condition that any two triangles x_1, x_2, x_3 and y_1, y_2, y_3 be directly similar is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0; \tag{1}$$

therefore, since the two triangles x, a, b and $\bar{y}$, $\bar{b}$, $\bar{a}$ are directly similar,

$$\begin{vmatrix} x & a & b \\ \bar{y} & \bar{b} & \bar{a} \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{2}$$

Expanding this determinant, we obtain

$$\frac{x}{(a-b)} + \frac{\bar{y}}{(\bar{a}-\bar{b})} = \frac{a\bar{a} - b\bar{b}}{(a-b)(\bar{a}-\bar{b})}, \tag{3}$$

which can be written more symmetrically, as

$$\frac{x-b}{a-b} + \frac{\bar{y}-\bar{b}}{\bar{a}-\bar{b}} = 1 \tag{4}$$

or

$$\frac{x-a}{a-b} + \frac{\bar{y}-\bar{a}}{\bar{a}-\bar{b}} = 1. \tag{5}$$

If $b = 0 =$, $\bar{b}$ (4) becomes

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1, \tag{6}$$

which is the standard equation for reflexion in a line.

2. *Map-equation of the Double Parabola.**

The map-equation of the double parabola is

$$x = \frac{A_1}{(\alpha_1 - t)^2} + \frac{A_2}{(\alpha_2 - t)^2} \tag{1}$$

(where α_1, α_2, t are turns or orthogonal numbers), provided the cusp condition $dx/dt = 0$ is satisfied for t a turn.

Applying the cusp condition to (1) we have

$$\frac{dx}{dt} = \sum^{2} \frac{2A_i}{(\alpha_i - t)^3} = 0. \tag{2}$$

The conjugate equation of (2) is

$$\sum^{2} \frac{2\bar{A}_i \alpha_i^3 t^3}{(\alpha_i - t)^3} = 0. \tag{$\bar{2}$}$$

* The name is due to Clifford.

Hence, the condition that (1) has a cusp is satisfied when

$$A_i = \bar{A}_i\alpha_i^3. \tag{3}$$

And, in general, the map-equation of an n-fold parabola is

$$x = \sum^n \frac{A_i}{(\alpha_i - t)^2}, \tag{4}$$

provided the cusp condition $dx/dt = 0$ is satisfied for a turn t, that is, when (3) is satisfied for $i = 1, 2, 3, \cdots, n$.

3. *The Equation of a Directed Line.*

Directed lines are lines which possess a right- and left-hand side, and therefore should be marked with an arrow head.

The normal form of a line, in rectangular coördinates, is

$$X \cos\alpha + Y \sin\alpha = p, \tag{1}$$

where p is the $\perp$ distance from the line to the origin.

If, now, we employ the circular coördinates $x = X + iY$, $\bar{x} = X - iY$, instead of the rectangular coördinates, (1) becomes

$$(x + \bar{x}) \cos\alpha + \left(\frac{x - \bar{x}}{i}\right) \sin\alpha = 2p \tag{2}$$

or

$$x(\cos\alpha - i\sin\alpha) + \bar{x}(\cos\alpha + i\sin\alpha) = 2p. \tag{3}$$

But $(\cos\alpha - i\sin\alpha)$ is a turn; let us represent it by t, then

$$(\cos\alpha + i\sin\alpha) = 1/t.$$

Putting these values in (3) we have

$$xt + \bar{x}/t = 2p \tag{4}$$

as the equation of a directed line.

4. *Distance from a Straight Line.*

In the case of directed lines, distance = length ÷ direction.

Let 0 be the reflexion of $a = 2p/t$ (where p is real) in the axis, then the equation of the axis is

$$yt + \bar{x}/t = 2p. \tag{1}$$

From the figure we see that the distance of the point x from the axis is

$$\frac{x - y}{2 \cdot 1/t}. \tag{2}$$

But from (1) we have that

$$t y = -\bar{x}/t + 2p. \tag{3}$$

Therefore,

$$\frac{x - y}{2 \cdot 1/t} = \tfrac{1}{2}(tx + \bar{x}/t - 2p). \tag{4}$$

Thus, twice the distance of a point x from a line is expressed by the equation

$$2D = tx + \bar{x}/t - 2p.$$

Undirected Lines.

§ 3. FIRST CHAIN OF THEOREMS.

1—*a*. Given three lines λ_1, λ_2 and λ_3 tangent to the parabola

$$x = \frac{1}{(1+t)^2} \tag{1}$$

at the points t_1, t_2 and t_3 respectively, then the equation of λ_i (for $i = 1, 2, 3$) will be

$$x = \frac{1}{(1+t_i)(1+t)}. \tag{2}$$

The intersection of λ_1 and λ_2 is given by the equation

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)}, \tag{3}$$

for if we let $t = t_2$ in the equation of λ_1, it is identical with (3); therefore (3) is the equation of a point on line λ_1; and similarly, if we let $t = t_1$, in the equation of λ_2, this equation becomes the same as (3); therefore (3) is the equation of a point on the line λ_2, consequently (3) is the equation of the intersection of the two lines λ_1 and λ_2. In like manner, we obtain the intersections of λ_2 and λ_3, and λ_3 and λ_1. Its conjugate equation is

$$\bar{x}_{12} = \frac{t_1 t_2}{(1+t_1)(1+t_2)}. \tag{4}$$

We shall now proceed to reflect these intersections x_{12}, x_{23}, x_{31} in an arbitrary line, say the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{5}$$

From section 1 we see that reflexion in the line is given by the equation

$$\frac{x}{a} + \frac{\bar{y}}{\bar{a}} = 1. \tag{6}$$

Therefore, reflecting the point x_{12} (the intersection of λ_1 and λ_2) whose coördinates are

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)}; \qquad \bar{x} = \frac{t_1 t_2}{(1+t_1)(1+t_2)}, \tag{7}$$

we obtain, as its reflexion in the line, the point

$$\frac{x}{a} = 1 - \frac{t_1 t_2}{\bar{a}(1+t_1)(1+t_2)}, \tag{8}$$

its conjugate

$$\frac{\bar{x}}{\bar{a}} = 1 - \frac{1}{a(1+t_1)(1+t_2)}. \tag{9}$$

In a like manner we can obtain the reflexions of the points x_{23} and x_{31} in the line.

The equation of a tangent to the parabola at the point t_1 is

$$x = \frac{1}{(1+t_1)(1+t)} \tag{10}$$

and its conjugate equation is

$$\bar{x} = \frac{t_1 t}{(1+t_1)(1+t)}. \tag{11}$$

Adding (10) and (11) we have

$$x + \bar{x} = \frac{1}{(1+t_1)(1+t)} + \frac{t_1 t}{(1+t_1)(1+t)}, \tag{12}$$

or

$$x + \frac{\bar{x}}{t_1} = \frac{1}{1+t_1}, \tag{13}$$

which is a self-conjugate equation of the tangent to the parabola at the point t_1. A line perpendicular to this tangent will be of the form

$$x - \frac{\bar{x}}{t_1} = c - \frac{\bar{c}}{t_1}. \tag{14}$$

Consequently, the equation of a line on the reflexion of x_{23} and perpendicular to the line λ_1 (which is tangent to the parabola at the point t_1) is

$$x - \frac{\bar{x}}{t_1} = a - \frac{a t_2 t_3}{\bar{a}(1+t_2)(1+t_3)} - \frac{1}{t_1}\left[\bar{a} - \frac{\bar{a}}{a(1+t_2)(1+t_3)}\right]. \tag{15}$$

For convenience let us rewrite (15) in the following manner:

$$t_1x - \bar{x} = at_1 - \frac{at_1t_2t_3(1+t_1)}{\bar{a}(1+t_1)(1+t_2)(1+t_3)} - \bar{a} + \frac{\bar{a}(1+t_1)}{a(1+t_1)(1+t_2)(1+t_3)} \qquad (16)$$

or

$$t_1x - \bar{x} = at_1 - \frac{as_3(1+t_1)}{\bar{a}\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}(1+t_1)}{a\overset{3}{\pi}}, \qquad (17)$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_3)$; $s_3 = t_1t_2t_3$.

Let us replace t_1 in (17) by the parameter t, and we obtain the equation

$$tx - \bar{x} = at - \frac{as_3(1+t)}{\bar{a}\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}(1+t)}{a\overset{3}{\pi}}. \qquad (18)$$

Now then, if we let $t = t_2$, it is easily seen that (18) becomes the equation of a line on the reflexion of x_{31} and perpendicular to the line λ_2; similarly when $t = t_3$, (18) represents the line on the reflexion of x_{12} and perpendicular to the line λ_3.

Thus, by means of a process called "interpolation" we are enabled to write one equation, which as the parameter "t" assumes the values t_1, t_2, t_3 picks up the three lines μ_1, μ_2 and μ_3. Therefore, the three lines μ_1, μ_2 and μ_3 must intersect in a point, and their point of intersection is expressed by the equation

$$x_{123} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}}. \qquad (19)$$

Hence we have the theorem of Desboves:

Given three lines and an extra line; if we reflect the vertices of the three lines in the arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the three perpendiculars will meet in a point.*

We will call this point the associated point of the line λ with reference to the three given lines λ_1, λ_2, λ_3.

1—*b*. Again, let us consider four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1+t)^2}. \qquad (1)$$

We will recall that λ_i (for $i = 1, 2, 3, 4$) is expressed by the equation

$$x = \frac{1}{(1+t_i)(1+t)}. \qquad (2)$$

For many purposes the circumcenter of a 3-line plays the part of the intersection of a 2-line. We will now find the circumcenter of three tangents to the parabola. First, let us examine the intersections of the 3-line formed by λ_1, λ_2, λ_3. On page 125 we defined the intersection of two tangents λ_i and λ_j as follows:

$$x_{ij} = \frac{1}{(1+t_i)(1+t_j)}. \tag{3}$$

Therefore, the intersections of λ_1, λ_2; λ_2, λ_3 and λ_3, λ_1 have as their respective equations

$$x_{12} = \frac{1}{(1+t_1)(1+t_2)}, \tag{4}$$

$$x_{23} = \frac{1}{(1+t_2)(1+t_3)}, \tag{5}$$

$$x_{31} = \frac{1}{(1+t_3)(1+t_1)}. \tag{6}$$

By means of interpolation we are enabled to write an equation that will pick up all three of these points, such an equation is

$$x = \frac{(1+t)}{(1+t_1)(1+t_2)(1+t_3)}, \tag{7}$$

for when $t = t_1$, (7) reduces to (5), and therefore (7) passes through the intersection of λ_2 and λ_3, that is, (7) is on the point x_{23}; when $t = t_2$, (7) reduces to (6), and is therefore on x_{31}; and finally when $t = t_3$, (7) reduces to (4) and passes through the point x_{12}. Consequently (7) represents a curve which passes through the three points x_{12}, x_{23} and x_{31}. (7) is the map-equation of a circle, whose center is

$$x_{123} = \frac{1}{(1+t_1)(1+t_2)(1+t_3)}. \tag{8}$$

and the conjugate equation is

$$\bar{x}_{123} = \frac{s_3}{(1+t_1)(1+t_2)(1+t_3)}. \tag{9}$$

This is then the circumcenter. Therefore, associated with four undirected lines, we will have four circumcenters, one for each 3-line.

We shall now proceed as we did in section 1—*a*, page 125; first we will reflect the circumcenter x_{123} in an arbitrary line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{10}$$

and then we will erect a perpendicular to λ_4 on this reflexion. Since reflexion in a line is given by

$$\frac{x}{a}+\frac{\bar{y}}{\bar{a}}=1 \tag{11}$$

we will obtain as the reflexion of x_{123} in the line (10)

$$\frac{x}{a}=1-\frac{1}{\bar{a}}\left[\frac{s_3}{(1+t_1)(1+t_2)(1+t_3)}\right] \tag{12}$$

and its conjugate equation

$$\frac{\bar{x}}{\bar{a}}=1-\frac{1}{a}\left[\frac{1}{(1+t_1)(1+t_2)(1+t_3)}\right]. \tag{13}$$

Now since λ_4 is a tangent to the parabola

$$x=\frac{1}{(1+t)^2} \tag{14}$$

at the point t_4, we can write its equation as

$$x+\frac{\bar{x}}{t_4}=\frac{1}{1+t_4}. \tag{15}$$

A line $\perp$ to (15) will be of the form

$$x-\frac{\bar{x}}{t_4}=c-\frac{\bar{c}}{t_4}. \tag{16}$$

Therefore, the equation of a line perpendicular to λ_4, and on the reflexion of the point x_{123}, in the arbitrary line

$$\frac{x}{a}+\frac{\bar{x}}{\bar{a}}=1, \tag{17}$$

will have the following equation, namely,

$$x-\frac{\bar{x}}{t_4}=a-\frac{as_3'}{\bar{a}\overset{3}{\pi}}-\frac{\bar{a}}{t_4}+\frac{\bar{a}}{a\overset{3}{\pi}t_4} \tag{18}$$

or

$$t_4x-\bar{x}=at_4-\frac{as_4}{\bar{a}\overset{3}{\pi}}-\bar{a}+\frac{\bar{a}}{a\overset{3}{\pi}}, \tag{19}$$

where $\overset{3}{\pi}=(1+t_1)(1+t_2)(1+t_3)$; $s_4=t_1t_2t_3t_4$ and $s_3'=t_1t_2t_3$. Similarly, the equation of a line perpendicular to λ_3, and on the reflexion of the point x_{124}, in the arbitrary line, is

$$x-\frac{\bar{x}}{t_3}=a-\frac{as_3'}{\bar{a}\overset{3}{\pi}}-\frac{\bar{a}}{t_3}+\frac{\bar{a}}{a\overset{3}{\pi}t_3} \tag{20}$$

or

$$t_3x - \bar{x} = at_3 - \frac{as_4}{a\overset{3}{\pi}} - \bar{a} + \frac{\bar{a}}{a\overset{3}{\pi}}, \tag{21}$$

where $\overset{3}{\pi} = (1 + t_1)(1 + t_2)(1 + t_4)$; $s_4 = t_1t_2t_3t_4$ and $s_3' = t_1t_2t_4$.

Again, by the process called "interpolation," we are enabled to write an equation that will pick up these four lines, namely,

$$tx - \bar{x} = at - \frac{as_4(1 + t)}{\bar{a}\overset{4}{\pi}} - \bar{a} + \frac{\bar{a}(1 + t)}{a\overset{4}{\pi}} \tag{22}$$

(where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1t_2t_3t_4$), for when $t = t_4$, (22) reduces to (19), and similarly when $t = t_3$, (22) reduces to (21), etc. Thus, it is easily seen that (22) picks up the four lines which are perpendicular to λ_1, λ_2, λ_3 and λ_4 respectively, and which also lie, respectively, on the reflexions of x_{234}, x_{134}, x_{124} and x_{123} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{23}$$

Moreover, (22) gives the intersection of these four perpendicular lines. Consequently, the point of intersection of the four lines is

$$x_{1234} = a - \frac{as_4}{\bar{a}\overset{4}{\pi}} + \frac{\bar{a}}{a\overset{4}{\pi}} \tag{24}$$

and its conjugate equation is

$$\bar{x}_{1234} = \bar{a} - \frac{\bar{a}}{a\overset{4}{\pi}} + \frac{as_4}{\bar{a}\overset{4}{\pi}},$$

where $\overset{4}{\pi} = (1 + t_1)(1 + t_2)(1 + t_3)(1 + t_4)$; $s_4 = t_1t_2t_3t_4$.

Hence we can state the following theorem:

Given four lines and an extra line, if we reflect the circumcenters of the four 3-lines in the arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the four perpendiculars will meet in a point.*

1—*c*. Now, we shall consider five undirected lines as the tangents of a double parabola. The equation of the double parabola is

$$x = \frac{A}{(\alpha - t)^2} + \frac{B}{(\beta - t)^2}, \tag{1}$$

provided the cusp condition $dx/dt = 0$; for t a turn. In this case the cusp condition is $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$.

Let the five lines under consideration be designated as λ_1, λ_2, λ_3, λ_4 and λ_5. The equation of a tangent to the double parabola at the point t_1 is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \tag{2}$$

or, in general, the equation of λ_i ($i = 1, 2, 3, 4, 5$) is

$$x = \frac{A}{(\alpha - t_i)(\alpha - t)} + \frac{B}{(\beta - t_i)(\beta - t)}. \qquad (3)$$

We shall now proceed to find the intersection of two tangents, say λ_1 and λ_2. Since the equation of λ_1 is

$$x = \frac{A}{(\alpha - t_1)(\alpha - t)} + \frac{B}{(\beta - t_1)(\beta - t)} \qquad (4)$$

and that of λ_2 is

$$x = \frac{A}{(\alpha - t_2)(\alpha - t)} + \frac{B}{(\beta - t_2)(\beta - t)}, \qquad (5)$$

then the equation of x_{12}, the intersection of λ_1, λ_2, will be

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)}. \qquad (6)$$

Similarly, we can find the intersections of x_{13}, x_{23}, x_{14}, x_{15}, etc.

In section 1—b, we found that to each 3-line there was associated a circumcenter; therefore to every 3-line arising from these five lines there is associated a circumcenter. Since we know the equation of the intersection of λ_1, λ_2, we can, by symmetry, write the equations of the intersections of λ_1, λ_3 and λ_2, λ_3. Thus

$$x_{12} = \frac{A}{(\alpha - t_1)(\alpha - t_2)} + \frac{B}{(\beta - t_1)(\beta - t_2)}, \qquad (7)$$

$$x_{23} = \frac{A}{(\alpha - t_2)(\alpha - t_3)} + \frac{B}{(\beta - t_2)(\beta - t_3)}, \qquad (8)$$

$$x_{31} = \frac{A}{(\alpha - t_3)(\alpha - t_1)} + \frac{B}{(\beta - t_3)(\beta - t_1)}. \qquad (9)$$

By means of interpolation we can get the equation of the circumcircle, which passes through the points x_{12}, x_{23}, x_{31}, for if $t = t_i$ (where $i = 1, 2, 3$) in the equation

$$x = \frac{A(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)}, \qquad (10)$$

(10) becomes successively (8), (9) and (7).

The center of the circle (10) is

$$x_{123} = \frac{A\alpha}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)} + \frac{B\beta}{(\beta - t_1)(\beta - t_2)(\beta - t_3)}. \qquad (11)$$

But from the four lines we can form four 3-lines, and since for every 3-line there is a circumcenter, therefore, from the four lines λ_1, λ_2, λ_3 and λ_4 we can obtain the four circumcenters x_{123}, x_{124}, x_{134}, x_{234}. From (11) we can write their equations, by means of symmetry.

Interpolating, we obtain

$$x = \frac{A\alpha(\alpha - t)}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta(\beta - t)}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}, \quad (12)$$

which equation, obviously, as t assumes successively the values t_1, t_2, t_3, t_4, picks up the points x_{123}, x_{124}, x_{134}, x_{234}.

But (12) is the equation of a circle, and its center is

$$x_{1234} = \frac{A\alpha^2}{(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)} + \frac{B\beta^2}{(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)}, \quad (13)$$

$$\bar{x}_{1234} = \frac{\bar{A}(1/\alpha^2)}{(1/\alpha - 1/t_1)(1/\alpha - 1/t_2)(1/\alpha - 1/t_3)(1/\alpha - 1/t_4)} + \frac{\bar{B}(1/\beta^2)}{(1/\beta - 1/t_1)(1/\beta - 1/t_2)(1/\beta - 1/t_3)(1/\beta - 1/t_4)}. \quad (14)$$

Applying the cusp condition $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$ and letting

$$P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4)$$

and

$$Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4),$$

we have

$$\bar{x} = \frac{\bar{A}\alpha^2 s_4}{P_4} + \frac{\bar{B}\beta^2 s_4}{Q_4} \quad (15)$$

$$= \frac{A s_4}{\alpha P_4} + \frac{B s_4}{\beta Q_4} \quad (16)$$

$$= \sum^2 \frac{A s_4}{\alpha P_4}, \quad (17)$$

where $s_4 = t_1 t_2 t_3 t_4$.

If now we reflect this point x_{1234} in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1, \quad (18)$$

we obtain, as its reflexion, the point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha P_4}, \tag{19}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_4}. \tag{20}$$

The equation of a tangent to the double parabola at the point t_5 is

$$x = \frac{A}{(\alpha - t_5)(\alpha - t)} + \frac{B}{(\beta - t_5)(\beta - t)} \tag{21}$$

and its conjugate equation is

$$\bar{x} = \frac{\bar{A}\alpha^2 tt_5}{(\alpha - t_5)(\alpha - t)} + \frac{\bar{B}\beta^2 tt_5}{(\beta - t_5)(\beta - t)}. \tag{22}$$

Since the cusp condition is $\bar{A}\alpha^3 = A$; $\bar{B}\beta^3 = B$, we can write (22) as

$$\frac{\bar{x}}{t_5} = \frac{At}{\alpha(\alpha - t_5)(\alpha - t)} + \frac{Bt}{\beta(\beta - t_5)(\beta - t)}. \tag{23}$$

Subtracting (23) from (21) we obtain a self-conjugate expression for the tangent to the double parabola at the point t_5, namely,

$$x - \frac{\bar{x}}{t_5} = \frac{A}{\alpha(\alpha - t_5)} + \frac{B}{\beta(\beta - t_5)} \tag{24}$$

or

$$x - \frac{\bar{x}}{t_5} = \sum^{2} \frac{A}{\alpha(\alpha - t_5)}. \tag{25}$$

A line perpendicular to this tangent and on the point $\bar{x}_{1234}$ is expressed by the equation

$$x + \frac{\bar{x}}{t_5} = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha P_4} + \frac{\bar{a}}{t_5} - \frac{\bar{a}}{a t_5} \sum^{2} \frac{A\alpha^2}{P_4} \tag{26}$$

or

$$t_5 x + \bar{x} = a t_5 - \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{\alpha P_4} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_4}, \tag{27}$$

where

$$P_4 = (\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4),$$
$$Q_4 = (\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4)$$

and $S_5 = t_1 t_2 t_3 t_4 t_5$. But in this case we will have five perpendicular lines, namely, l_1 perpendicular to λ_1 and on $\bar{x}_{2345}$; $l_2 \perp \lambda_2$ and on $\bar{x}_{1345}$; $l_3 \perp \lambda_3$ and on $\bar{x}_{1245}$; $l_4 \perp \lambda_4$ and on $\bar{x}_{1235}$, and $l_5 \perp \lambda_5$ and on $\bar{x}_{1234}$.

Symmetrizing so as to pick up these five lines, we obtain the equation

$$tx + \bar{x} = at - \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{\alpha P_5} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{P_5} . \tag{28}$$

It is at once evident, when $t = t_i$ (where $i = 1, 2, 3, 4, 5$), (28) gives us the line l_i ($i = 1, 2, 3, 4, 5$).

But (28) is the equation of the intersection of the five lines l_1, l_2, l_3, l_4 and l_5. This point has the equation

$$x_{12345} = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{P_5} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{P_5} . \tag{29}$$

Consequently, we have the theorem:

Given five lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, the five perpendiculars will meet in a point.*

In general, we can say:

Given n lines and an extra line; if we reflect the centric points of the n "$(n - 1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, the n perpendiculars will meet in a point.

SECOND CHAIN OF THEOREMS.

2—*a*. Given four undirected lines λ_1, λ_2, λ_3 and λ_4 as tangents to the parabola

$$x = \frac{1}{(1 + t)^2} \tag{1}$$

at the points t_1, t_2, t_3 and t_4 respectively, then the equation of λ_i ($i = 1, 2, 3, 4$) is

$$x = \frac{1}{(1 + t_i)(1 + t)} . \tag{2}$$

With four undirected lines we can form four 3-lines for $C^4_3 = 4$. Here, the four 3-lines are composed of $\lambda_1, \lambda_2, \lambda_3$; $\lambda_1, \lambda_2, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4$; and $\lambda_2, \lambda_3, \lambda_4$ respectively. Now, by the theorem of section 1—*a* we found that to every 3-line $\lambda_i, \lambda_j, \lambda_k$ there is associated a point x_{ijk}. Consequently, to the 3-line $\lambda_1, \lambda_2, \lambda_3$ is associated a point x_{123}; to the 3-line $\lambda_1, \lambda_2, \lambda_4$ the point x_{124}, and so forth.

The equation of the point x_{123} is

$$x_{123} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} , \tag{3}$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_3)$; $s_3 = t_1t_2t_3$ and its conjugate equation is

$$\bar{x}_{123} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}. \tag{4}$$

Similarly, the equation of the point x_{124} is

$$x_{124} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \tag{5}$$

and

$$\bar{x}_{124} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}, \tag{6}$$

where $\overset{3}{\pi} = (1+t_1)(1+t_2)(1+t_4)$; $s_3 = t_1t_2t_4$. And so, in general, we have, as the equation of x_{ijk}, the point associated with the 3-line $\lambda_i, \lambda_j, \lambda_k$:

$$x_{ijk} = a - \frac{as_3}{\bar{a}\overset{3}{\pi}} + \frac{\bar{a}}{a\overset{3}{\pi}} \tag{7}$$

and

$$\bar{x}_{ijk} = \bar{a} - \frac{\bar{a}}{a\overset{3}{\pi}} + \frac{as_3}{\bar{a}\overset{3}{\pi}}, \tag{8}$$

where $\overset{3}{\pi} = (1+t_i)(1+t_j)(1+t_k)$; $s_3 = t_it_jt_k$.

Symmetrizing so as to pick up the four points x_{123}, x_{124}, x_{134}, x_{234}, we obtain the equation

$$x = a - \frac{as_4(1+t)}{\bar{a}t\overset{4}{\pi}} + \frac{\bar{a}(1+t)}{a\overset{4}{\pi}} \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}(1+t)}{a\overset{4}{\pi}} + \frac{as_4(1+t)}{\bar{a}t\overset{4}{\pi}}, \tag{10}$$

where $\overset{4}{\pi} = (1+t_1)(1+t_2)(1+t_3)(1+t_4)$; $s_4 = t_1t_2t_3t_4$. Adding (9) and (10) we have

$$x + \bar{x} = a + \bar{a},$$

which is the equation of a vertical line.

Consequently, we have another theorem, namely:

Given four lines and an extra line λ, the four associated points, which arise from the four 3-lines and an arbitrary line λ, lie on a line.

Cwojdzinski states this theorem.*

2—*b*. Given five lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^3} \tag{1}$$

* *Archiv der Mathematik und Physik*, Vol. 1, 1901, p. 180.

(cusp condition being $\bar{A}_i\alpha_i^3 = A_i$), at the points t_1, t_2, t_3, t_4 and t_5 respectively, then the equation of the line λ_i $(i = 1, 2, 3, 4, 5)$ is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)} . \tag{2}$$

With five undirected lines we can form five 4-lines for $C_4^5 = 5$. Here the five 4-lines are composed of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; $\lambda_1, \lambda_3, \lambda_4, \lambda_5$; $\lambda_1, \lambda_2, \lambda_4, \lambda_5$; $\lambda_1, \lambda_2, \lambda_3, \lambda_5$; $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively. Now, by theorem 1—*b* we found that to every 4-line $\lambda_i, \lambda_j, \lambda_k, \lambda_l$ there is associated a point x_{ijkl}. Consequently, to the above-named 4-lines are associated the points x_{1234}, x_{1345}, x_{1235}, x_{1245}, x_{2345}. The equation of an associated point given in section 1—*b* was for four lines taken as tangents to a parabola with the arbitrary line λ; here it will be necessary to find the equation of the associated point, the four lines being taken as tangents to a double parabola.

From 1—*c* we have the equation of the circumcenter for three lines $\lambda_1, \lambda_2, \lambda_3$ taken as tangents to a double parabola, namely,

$$x_{123} = \sum^{2} \frac{A\alpha}{\overset{3}{\pi}} , \tag{3}$$

and its conjugate is

$$\bar{x}_{123} = - \sum^{2} \frac{As_3}{\alpha\overset{3}{\pi}} . \tag{4}$$

Reflecting this point in the line

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1 \tag{5}$$

we obtain the point

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_3}{\alpha\overset{3}{\pi}} , \tag{6}$$

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{3}{\pi}} . \tag{7}$$

A line $\perp$ to the tangent λ_4 and on the reflexion of the point x_{123} in the arbitrary line λ is

$$t_4 x + \bar{x} = t_4 a + \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha\overset{3}{\pi}} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{3}{\pi}} . \tag{8}$$

Now interpolating so as to pick up these four perpendiculars, we obtain the equation

$$tx + \bar{x} = ta + \frac{a}{\bar{a}} \sum^{2} \frac{As_4(\alpha - t)}{\alpha\overset{4}{\pi}} + \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha(\alpha - t)}{\overset{4}{\pi}} , \tag{9}$$

which is an associated point

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_4}{\alpha \overset{4}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha}{\overset{4}{\pi}}, \tag{10}$$

where $\overset{4}{\pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)$; $s_4 = t_1 t_2 t_3 t_4$.

Interpolating so as to pick up these five associated points, we have

$$x = a - \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{\alpha t \overset{5}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha(\alpha - t)}{\overset{5}{\pi}}. \tag{11}$$

This is the equation of an ellipse, and its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{\overset{5}{\pi}} + \frac{a}{\bar{a}} \sum^{2} \frac{As_5(\alpha - t)}{t \overset{5}{\pi}}, \tag{12}$$

where $\overset{5}{\pi} = (\alpha_i - t_1)(\alpha_i - t_2)(\alpha_i - t_3)(\alpha_i - t_4)(\alpha_i - t_5)$; $s_5 = t_1 t_2 t_3 t_4 t_5$.

Hence we have the theorem:

Given five lines and an extra line λ, the five associated points, which arise from the five 4-lines, and an arbitrary line λ, lie on an ellipse.

2—*c*. Given six lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ as tangents to the double parabola

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t)^2} \tag{1}$$

(cusp condition being $\bar{A}_i \alpha_i^3 = A$), at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively, then the equation of the tangent λ_i ($i = 1, 2, 3, 4, 5, 6$) is

$$x = \sum^{2} \frac{A_i}{(\alpha_i - t_i)(\alpha_i - t)}. \tag{2}$$

With six undirected lines we can form six 5-lines for $C_5^6 = 6$. From theorem 1—*c* we found that to every 5-line and an arbitrary line there is associated a point. Therefore to six lines and an arbitrary line there will be associated six points. The equation of the point x_{12345} is

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_5}{\alpha \overset{5}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2}{\overset{5}{\pi}}. \tag{3}$$

Symmetrizing so as to pick up the six points, namely, x_{12345}, x_{12346}, x_{12456}, x_{13456}, x_{12356}, x_{23456}, we have

$$x = a + \frac{a}{\bar{a}} \sum^{2} \frac{As_6(\alpha - t)}{\alpha t \overset{6}{\pi}} + \frac{\bar{a}}{a} \sum^{2} \frac{A\alpha^2(\alpha - t)}{\overset{6}{\pi}}, \tag{4}$$

which is the map-equation of an ellipse. Its conjugate equation is

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\sum^{2}\frac{A\alpha^3(\alpha - t)}{\overset{6}{\pi}} - \frac{a}{\bar{a}}\sum^{2}\frac{As_6(\alpha - t)}{t\overset{6}{\pi}}\,. \tag{5}$$

Hence the theorem:

Given six lines and an extra line λ; the six associated points, which arise from the six 5-lines and an arbitrary line λ, lie on an ellipse.

2—*g*. In general, we can say:

Given n lines and an arbitrary line λ; the n associated points, which arise from the n "$(n-1)$-lines" and an arbitrary line λ, lie on an ellipse.

Directed Lines.

FIRST CHAIN OF THEOREMS.

I—*a*. Suppose we have four directed lines L_1, L_2, L_3, L_4 given as tangents to a parastroid at the points t_1, t_2, t_3 and t_4; then since the equation of the parastroid is

$$t^4 - t^3x + \mu t^2 - t\bar{x} + 1 = 0 \tag{1}$$

(where μ is real), the equations of L_1, L_2, L_3, L_4 will be respectively

$$t_1^4 - t_1^3x + \mu t_1^2 - t_1\bar{x} + 1 = 0, \tag{2}$$
$$t_2^4 - t_2^3x + \mu t_2^2 - t_2\bar{x} + 1 = 0, \tag{3}$$
$$t_3^4 - t_3^3x + \mu t_3^2 - t_3\bar{x} + 1 = 0, \tag{4}$$
$$t_4^4 - t_4^3x + \mu t_4^2 - t_4\bar{x} + 1 = 0. \tag{5}$$

The incenter* of the lines L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3} \tag{6}$$

and its conjugate is

$$\bar{x}_{123} = 1/t_1 + 1/t_2 + 1/t_3 + t_1t_2t_3. \tag{7}$$

The reflexion of the point x_{123} in the line λ, namely,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1, \tag{8}$$

is

$$x = a - \frac{a}{\bar{a}}[1/t_1 + 1/t_2 + 1/t_3 + t_1t_2t_3] \tag{9}$$

and

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}\left[t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right]. \tag{10}$$

We now desire to drop a perpendicular from the reflexion of x_{123} to the

* Hodgson, Joseph E., "Orthocentric Properties of the Plane Directed n-Line," *Transactions of the American Mathematical Society*, Vol. 13, 1912, p. 199.

line L_4; since a perpendicular to L_4 is of the form

$$t_4^2 x - \bar{x} = t_4^2 c - \bar{c}, \tag{11}$$

in order that this perpendicular pass through the reflexion of x_{123} in the line λ, (11) becomes

$$t_4^2 x - \bar{x} = t_4^2\left[a - \frac{a}{\bar{a}}\left(\frac{s_2}{s_3} + s_3\right)\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(s_1 + \frac{1}{s_3}\right)\right]. \tag{12}$$

In order to interpolate, so as to be able to pick up the four perpendiculars, namely, the one from the reflexion of x_{234} to L_1; from the reflexion of x_{134} to L_2, etc., we will introduce the following symmetric functions:

$$\begin{aligned}
s_1 &= t_1 + t_2 + t_3, & \sigma_1 &= t_1 + t_2 + t_3 + t_4,\\
s_2 &= t_1t_2 + t_1t_3 + t_2t_3, & \sigma_2 &= t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4,\\
s_3 &= t_1t_2t_3, & \sigma_3 &= t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4,\\
& & \sigma_4 &= t_1t_2t_3t_4;
\end{aligned}$$

$$\begin{aligned}
\therefore\quad s_1 &= \sigma_1 - t_4,\\
s_2 &= \sigma_2 - t_4\sigma_1 - t_4^2,\\
s_3 &= \sigma_4/t_4.
\end{aligned}$$

Now then, making use of the above symmetric functions, (12) can be written as

$$t_4^2 x - \bar{x} = t_4^2 a - \frac{a}{\bar{a}}\left[\frac{(t_4^2\sigma_2 - t_4^3\sigma_1 + t_4^4)t_4}{\sigma_4} + \frac{t_4\sigma_4^2}{\sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(s_1 + \frac{1}{s_3}\right)\right] \tag{13}$$

But

$$t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0, \tag{14}$$

$$\therefore\quad t_4 - \sigma_1 t^3 + \sigma_2 t^2 = \sigma^3 t - \sigma_4. \tag{15}$$

Making use of this relation in (13), we obtain the equation

$$t_4^2 x - \bar{x} = t_4^2 a - \left[\frac{at_4^2\sigma_3 - at_4\sigma_4 + at_4\sigma_4^2}{\bar{a}\sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(s_1 + \frac{1}{s_3}\right)\right]. \tag{16}$$

Thus (16) is the equation of a line which passes through the reflexion of x_{123}, and is perpendicular to L_4; which for convenience we will call R_4. There will be four such lines, namely, R_1, R_2, R_3, R_4, which will be obtained from the original four directed lines L_1, L_2, L_3 and L_4.

Symmetrizing so as to pick up these four lines, we have

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2\sigma_3 - at\sigma_4 + at\sigma_4^2}{\bar{a}\sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(\sigma_1 - t + \frac{t}{\sigma_4}\right)\right], \tag{17}$$

which is a circle.

As a result, we have the theorem:

Given four directed lines and an extra line λ; if we reflect the incenters of the four 3-lines in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these four perpendiculars will touch a circle.

I—*b*. We will next consider the five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4 and t_5 respectively. L_i (where $i = 1, 2, 3, 4, 5$) will be expressed by the equation

$$t_i^4 - t_i^3 x + \mu t_i^2 - t_i \bar{x} + 1 = 0, \tag{1}$$

where μ is real.

As given in section I—*a*, the incenter of the 3-line L_1, L_2, L_3 is

$$x_{123} = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}. \tag{2}$$

If now, instead of taking three directed lines, we take four directed lines, namely, L_1, L_2, L_3, L_4, then we will be able to form four 3-lines, and since there is an incenter associated with each 3-line, there will be four incenters associated with the four directed lines. Interpolating, so as to pick up these four incenters, we obtain the equation

$$x = \sigma_1 - t + \frac{t}{\sigma_4}, \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4$ and $\sigma_4 = t_1 t_2 t_3 t_4$. Now (3) is the equation of a circle whose center is

$$x_{1234} = \sigma_1. \tag{4}$$

We will now proceed to reflect this center in an arbitrary line, say,

$$\frac{x}{a} + \frac{\bar{x}}{\bar{a}} = 1. \tag{5}$$

Doing this, we obtain as the reflexion of x_{1234}

$$x = a - \frac{a}{\bar{a}}\left[\frac{\sigma_3}{\sigma_4}\right] \tag{6}$$

and its conjugate equation

$$\bar{x} = \bar{a} - \frac{\bar{a}}{a}[\sigma_1]. \tag{7}$$

As in the previous cases, we desire to drop a perpendicular from the reflexion of x_{1234} to the remaining line L_5. The equation of such a perpendicular is

$$t_5^2 x - \bar{x} = t_5^2\left[a - \frac{a}{\bar{a}}\left(\frac{\sigma_3}{\sigma_4}\right)\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right]. \tag{8}$$

But in this case, since we have five directed lines, we will obtain five perpendiculars similar to the one expressed by equation (8). In order to interpolate we will have to introduce the following symmetric functions:

$$
\begin{aligned}
\sigma_1 &= t_1 + t_2 + t_3 + t_4, & P_1 &= \sum^{5} t_1,\\
\sigma_2 &= \sum^{6} t_1 t_2, & P_2 &= \sum^{10} t_1 t_2,\\
\sigma_3 &= \sum^{4} t_1 t_2 t_3, & P_3 &= \sum^{10} t_1 t_2 t_3,\\
\sigma_4 &= t_1 t_2 t_3 t_4, & P_4 &= \sum^{5} t_1 t_2 t_3 t_4,\\
& & P_5 &= t_1 t_2 t_3 t_4 t_5;
\end{aligned}
$$

$$
\begin{aligned}
\therefore\quad \sigma_1 &= P_1 - t_5,\\
\sigma_2 &= P_2 - P_1 t_5 + t_5^2,\\
\sigma_3 &= P_3 - t_5 \sigma_2,\\
\sigma_4 &= P_5 / t_5.
\end{aligned}
$$

But since the "Ps" can be considered as the symmetric functions of a quintic in t, we obtain the relation

$$t^5 - P_1 t^4 + P_2 t^3 - P_3 t^2 + P_4 t - P_5 = 0, \tag{9}$$

from which we have

$$t_5^2 \sigma_3 = t_5^2 P_3 - t_5^3 \sigma_2 = t_5^2 P_3 - t_5^3 P_2 + t_5^4 P_1 - t_5^5 \tag{10}$$

and also that

$$t_5^2 P_3 - t_5^3 P_2 + t_5^4 P_1 - t_5^5 = t_5 P_4 - P_5. \tag{11}$$

Making use of the relations that exist among these two sets of symmetric functions, (8) can be written as

$$t_5^2 x - \bar{x} = t_5^2 a - \frac{a}{\bar{a}}\left[\frac{(t_5^2 P_3 - t_5^3 \sigma_2) t_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right] \tag{12}$$

or applying (11) to (12) we obtain

$$t_5^2 x - \bar{x} = t_5^2 a - \frac{a}{\bar{a}}\left[\frac{P_4 t_5^2 - P_5 t_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(\sigma_1)\right]. \tag{13}$$

Interpolating so as to pick up the five perpendiculars, we have the equation

$$t^2 x - \bar{x} = t^2 a - \frac{a}{\bar{a}}\left[\frac{t^2 P_4 - t P_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(P_1 - t)\right], \tag{14}$$

which is a circle.

Consequently, we have the following theorem:

Given five directed lines and an extra line; if we reflect the centric points of the five 4-lines in an arbitrary line λ, *and then drop perpendiculars from these reflexions to the remaining line, these five perpendiculars will touch a circle.*

I—g. In general, we can say:

Given n directed lines and an extra line; if we reflect the centric points of the n "$(n-1)$-lines" in an arbitrary line λ, and then drop perpendiculars from these reflexions to the remaining line, these n perpendiculars will touch a circle.

SECOND CHAIN OF THEOREMS.

II—a. Let us consider five directed lines L_1, L_2, L_3, L_4, L_5 as tangents to a parastroid, at the points t_1, t_2, t_3, t_4, t_5 respectively; then L_i (where $i = 1, 2, 3, 4, 5$) can be expressed as

$$t_i^4 - t_i^3 x + t_i^2 \mu - t_i \bar{x} + 1 = 0, \tag{1}$$

where μ is real.

By the previous theorem I—a we found that to every four directed lines and an extra line λ there is associated a circle, and since there are five directed lines, we will have five 4-lines and, therefore, five circles associated with the five directed lines. The equation of one of these circles is

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2\sigma_3 - at\sigma_4 + at\sigma_4^2}{\bar{a}\sigma_4}\right] - \left[\bar{a} - \frac{\bar{a}}{a}\left(\sigma_1 - t + \frac{t}{\sigma_4}\right)\right]. \tag{2}$$

Symmetrizing so as to pick up the five circles, we obtain

$$\begin{aligned} t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} - \frac{1}{T}\right)\right] + \frac{at}{\bar{a}} - \frac{at\sigma_5}{\bar{a}T} \\ - \bar{a} + \frac{\bar{a}}{a}\left[\sigma_1 - t - T + \frac{tT}{\sigma_5}\right], \end{aligned} \tag{3}$$

where

$$\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5;$$
$$\sigma_5 = t_1 t_2 t_3 t_4 t_5.$$

For convenience, let us denote σ_5 by T, then (3) becomes

$$\begin{aligned} t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\sigma_4}{T} - \frac{1}{T}\right)\right] + \frac{at}{\bar{a}} - \frac{atT}{\bar{a}T} - \bar{a} \\ + \frac{\bar{a}}{a}\left[\sigma_1 - t - T - \frac{tT}{T}\right] \end{aligned} \tag{4}$$

or

$$t^2 x - \bar{x} = t^2 a - \left[\frac{at^2}{\bar{a}}\left(\frac{\sigma_4}{T} - \frac{1}{T}\right)\right] - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - T]. \tag{5}$$

Now then let $T = t$, and (5) becomes

$$t^2x - \bar{x} = t^2a - \left[\frac{at(\sigma_4 - 1)}{\bar{a}}\right] - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - t] \cdot \tag{6}$$

or

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - a^2t(\sigma_4 - 1) - \bar{a}^2a + \bar{a}^2(\sigma_1 - t), \tag{7}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - a^2t\sigma_4 + a^2t - \bar{a}^2t + \bar{a}^2\sigma_1 - \bar{a}^2a, \tag{8}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 - t(a^2\sigma_4 - a^2 + \bar{a}^2) + \bar{a}^2(\sigma_1 - a), \tag{9}$$

$$t^2x - \bar{x} = at^2 - \frac{t(a^2\sigma_4 - a^2 + \bar{a}^2)}{a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{a} \cdot \tag{10}$$

But $t = T = \sigma_5$,

$$\therefore \quad x - \frac{\bar{x}}{\sigma_5^2} = a - \frac{(a^2\sigma_4 - a^2 + \bar{a}^2)}{\sigma_5 a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{\sigma_5^2 a}; \tag{11}$$

this is a self-conjugate equation, and is therefore the equation of a line, the line which touches the five circles arising from the five 4-lines.

Hence we have the theorem:

Given five directed lines and an extra line; the five circles, which arise from the five 4-lines and an arbitrary line λ, *all touch the same line.*

II—*b*. Let us consider six directed lines $L_1, L_2, L_3, L_4, L_5, L_6$ as tangents to a cyclogen, at the points $t_1, t_2, t_3, t_4, t_5, t_6$ respectively; then L_i ($i = 1, 2, 3, 4, 5, 6$) can be expressed as

$$t_i^6 - ct_i^5 + xt_i^4 - \mu t_i^3 + \bar{x}t_i^2 - \bar{c}t_i + 1 = 0, \tag{1}$$

where $i = 1, 2, 3, 4, 5, 6$ and μ is real.

By a previous theorem I—*b* we found out that to every five directed lines and an arbitrary line λ there is associated a circle, and since there are six 5-lines which arise from six directed lines, there will be six circles associated with six directed lines.

The equation of one of these circles is

$$t^2x - \bar{x} = t^2a - \frac{a}{\bar{a}}\left[\frac{t^2P_4 - tP_5}{P_5}\right] - \left[\bar{a} - \frac{\bar{a}}{a}(P_1 - t)\right] \cdot \tag{2}$$

Symmetrizing so as to pick up these six circles, we obtain the equation

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \frac{1}{t_5} + \frac{1}{t_6} - \frac{1}{T}\right] + \frac{at}{\bar{a}} - \bar{a} + \frac{\bar{a}}{a}[\sigma_1 - t - T], \tag{3}$$

where $\sigma_1 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$; $\sigma_6 = t_1t_2t_3t_4t_5t_6$. For convenience,

let us denote σ_6 by T, then (3) becomes

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{at}{\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}t}{a} - \frac{\bar{a}T}{a}, \tag{4}$$

$$t^2x - \bar{x} = t^2a - \frac{at^2}{\bar{a}}\left[\frac{\sigma_5}{T} - \frac{1}{T}\right] + \frac{a^2t}{a\bar{a}} - \bar{a} + \frac{\bar{a}\sigma_1}{a} - \frac{\bar{a}^2t}{a\bar{a}} - \frac{\bar{a}T}{a}. \tag{5}$$

Now then, let $T = t$, and (5) becomes

$$t^2x - \bar{x} = t^2a - \left[\frac{at}{\bar{a}}(\sigma_5 - 1)\right] + \frac{a^2t}{a\bar{a}} - \frac{\bar{a}^2t}{a\bar{a}} - \bar{a} + \frac{\bar{a}}{a}(\sigma_1 - t), \tag{6}$$

$$a\bar{a}(t^2x - \bar{x}) = t^2a^2\bar{a} - a^2t(\sigma_5 - 1) + a^2t - \bar{a}^2t - a\bar{a}^2 + \bar{a}^2(\sigma_1 - 1), \tag{7}$$

$$a\bar{a}(t^2x - \bar{x}) = a^2\bar{a}t^2 + t(2a^2 - 2\bar{a}^2 - a^2\sigma_5) + \bar{a}^2(\sigma_1 - a), \tag{8}$$

$$t^2x - \bar{x} = at^2 + \frac{t(2a^2 - 2\bar{a}^2 - a^2\sigma_5)}{a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{a}. \tag{9}$$

But $t = T = \sigma_6$,

$$\therefore\ x - \frac{\bar{x}}{\sigma_6^2} = a + \frac{(2a^2 - 2\bar{a}^2 - a^2\sigma_5)}{\sigma_6 a\bar{a}} + \frac{\bar{a}(\sigma_1 - a)}{\sigma_6^2 a}; \tag{10}$$

this is a self-conjugate equation, and is therefore the equation of a line, the line which touches the six circles arising from the six 5-lines.

Consequently we have the theorem:

Given six directed lines and an extra line; the six circles, which arise from the six 5-lines and an arbitrary line λ, all touch the same line.

And, in general, we can say:

II—c. Given n directed lines and an extra line; the n circles, which arise from the n "$(n - 1)$-lines" and an arbitrary line λ, all touch the same line.

A PORISTIC SYSTEM OF EQUATIONS.

By L. B. Robinson.

In an attempt to reduce a certain system of partial differential equations to a canonical form, the author has met with some systems of poristic equations of which the following system is a simple type:

$$P_{rs} \equiv \sum_{k=0}^{n} \sum_{i=0}^{n} p_{ki} x_{kr} x_{is} = 0 \quad (r, s = 1, 2, \cdots, n). \tag{1}$$

$p_{ki} = p_{ik}$, $P_{rs} = P_{sr}$, the matrix $\|x_{kr}\|$ is of rank n.

These equations imply that n points on the quadric spread

$$Q \equiv \sum_{k=0}^{n} \sum_{i=0}^{n} p_{ki} x_k x_i = 0$$

are conjugate in pairs. This is true only when the quadric spread consists of two hyperplanes; consequently we infer that the equations (1) form a poristic set possessing no relevant solution at all unless certain conditions are satisfied and an infinite number of solutions when the conditions are satisfied. It is clear, in fact, that when the quadric spread does consist of two hyperplanes a set of n points on one hyperplane satisfies the requirements.

The case $n = 2$ is discussed by Clebsch,* and his method can be extended so as to cover the general case. It may be worth while to indicate the necessary analysis, as it can be extended to the case of a form of the fourth order; also because the conditions are, as M. Janet has pointed out, those under which the differential equation

$$\sum_{k=0}^{n} \sum_{i=0}^{n} p_{ki} \frac{\partial^2 u}{\partial x_k \partial x_i} = 0$$

can be reduced to the form

$$\frac{\partial}{\partial \xi_0} \sum_{m=1}^{n} \alpha_m \frac{\partial u}{\partial \xi_m} = 0.$$

Let us denote the determinant $|x_{ij}|$ $(i, j = 0, 1, 2, \cdots, n)$ by the symbol D, the minor of x_{00} in this determinant by the symbol Δ_{00}, and the minor of x_{ij} in the determinant Δ_{00} by the symbol M_{ij} $(i, j = 1, 2, \cdots, n)$. We shall also write

$$\sum_{s=1}^{n} x_{0s} M_{is} \equiv \Delta_{i0}.$$

* "Vorlesungen über Geometrie," Bd. I (new edition), p. 136.

We now proceed to reduce the system of equations (1) to a canonical form. With this end in view, we must write the sum

$$S^{(iktl)} \equiv 2\sum_{r=1}^{n} x_{kr}x_{ir}M_{tr}M_{lr} + \sum_{r=1}^{n}\sum_{s=1}^{n}(x_{kr}x_{is} + x_{ks}x_{ir}) \quad \begin{pmatrix} k, l, t = 1, 2, \cdots, n \\ i = 0, 1, 2, \cdots, n \end{pmatrix},$$

where $r > s$ in the double summation, and $i < k$, $t \gtrless l$ throughout. The above sum falls into two parts

$$\begin{aligned} S_1^{(iktl)} &\equiv \sum_{r=1}^{n} x_{kr}x_{ir}M_{tr}M_{lr} + \sum_{r=1}^{n}\sum_{s=1}^{n}[x_{kr}x_{is}M_{tr}M_{ls} + x_{ks}x_{ir}M_{ts}M_{lr}] \\ &\equiv \sum_{r=1}^{n} x_{kr}M_{tr}\sum_{s=1}^{n} x_{is}M_{ls} \end{aligned}$$

and $S_2^{(iktl)}$ obtained from $S_1^{(iktl)}$ by interchanging t and l. If $i \neq l$ and $i \neq 0$, the sum $\sum_{s=1}^{n} x_{is}M_{ls}$ must vanish, because it is then equal to a determinant two of whose columns are identical. Also if $k \neq t$ and $k \neq 0$ the sum $\sum_{r=1}^{n} x_{kr}M_{tr}$ vanishes for the same reason. Since $i < k$ we must assume that k is different from 0.

If we assume that $i = 0$, $k \neq 0$,

$$S_1 \equiv \sum_{r=1}^{n} x_{kr}M_{tr}\sum_{s=1}^{n} x_{0s}M_{ls}.$$

Furthermore if $k \neq t$, S_1 vanishes identically. Therefore we shall consider only the expression

$$S_1 \equiv \sum_{r=1}^{n} x_{tr}M_{tr}\sum_{s=1}^{n} x_{0,s}M_{ls},$$

where $k = t$. But

$$\sum_{r=1}^{n} x_{tr}M_{tr} \equiv \Delta_{00}, \qquad \sum_{s=1}^{n} x_{0,s}M_{ls} \equiv \Delta_{l0},$$

$$\therefore \quad S_1 \equiv \Delta_{l0}\Delta_{00}.$$

We now consider S_2. Interchange the rôle of t and l, and clearly it reduces to $S_2 \equiv \Delta_{t0}\Delta_{00}$.

One case only remains for consideration. Let $i = t$, $k = l$. Then $S_2 = \Delta_{00}^2$, while $S_1 = 0$. We cannot interchange t and l in this case for $i < k$ and $t \leqq l$.

We have assumed while discussing S that $i < k$. But we shall also be obliged to consider the set $S^{(iitl)}$ obtained from S by dividing it by 2 and assuming that $i = k$. The result is equal to the product

$$\sum_{r=1}^{n} x_{ir}M_{tr}\sum_{s=1}^{n} x_{is}M_{ls},$$

whose first factor vanishes unless $i = t$ or $i = 0$. In the first case, if $i > l$, the second factor reduces to a determinant, two of whose columns are equal; therefore the product vanishes. If $i = l$, the product becomes

$$\sum_{r=1}^{n} x_{ir} M_{tr} \sum_{s=1}^{n} x_{is} M_{ls} \equiv \Delta_{00}^2.$$

In the second case, the product becomes

$$\sum_{r=1}^{n} x_{0r} M_{tr} \sum_{s=1}^{n} x_{0s} M_{ls} \equiv \Delta_{l0}\Delta_{t0}.$$

In other words $\frac{1}{2}S^{(00tl)} \equiv \Delta_{l0}\Delta_{t0}$.

We next construct the sum

$$U \equiv \sum_{r=1}^{n} M_{tr} M_{lr} P_{rr} + \sum_{r=1}^{n}\sum_{s=1}^{n} (M_{tr}M_{ls} + M_{ts}M_{lr})P_{rs},$$

where $r > s$ in the double summation. From what has been previously written we see that

$$U \equiv \sum_{j=1}^{n} p_{jj} \frac{S^{(jjtl)}}{2} + p_{00}\frac{S^{(00tl)}}{2} + \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ki} S^{(iktl)} + \sum_{k=1}^{n} p_{k0} S^{(0ktl)}.$$

We assume first that $t \neq l$. Then, as we have seen, $\frac{1}{2}S^{(jjtl)}$ vanishes, unless $j = 0$, when it is equivalent to Δ_{00}^2. As for the third member of U, its elements vanish unless $i = t$, $k = l$, when we obtain

$$p_{lt} S^{(tltl)} \equiv \Delta_{00}^2 p_{lt}.$$

The fourth member vanishes unless $k = l$ or $k = t$. So we can write down

(2) $\quad U \equiv \Delta_{00}^2 p_{lt} + \Delta_{t0}\Delta_{l0} p_{00} + \Delta_{l0}\Delta_{00} p_{t0} + \Delta_{t0}\Delta_{00} p_{l0} \quad (l, t = 1, 2, \cdots, n).$

When $t = l$, we have seen that all the terms of the first summation vanish save

$$p_{ll}\frac{S^{(llll)}}{2} = \Delta_{00} p_{ll}.$$

All the terms of the third summation vanish unless $i = k = l$; but this cannot happen since $i < k$. We see readily that (2) becomes

$$(2') \qquad U \equiv \Delta_{00}^2 p_{ll} + \Delta_{l0}^2 p_{00} + 2\Delta_{l,0}\Delta_{00} p_{l,0}.$$

The set formed by the union of (2) and (2′) may be called the canonical form of system (1), because what solves one system clearly solves the other. The relation known as "correlation multiplicatoire"* exists between (2) and (2′).

* Riquier, "Les systèmes d'équations aux dérivées partielles," p. 255.

Write the polynomial

$$(3)\quad (p_{12}p_{13} - p_{12}p_{23})(p_{12}p_{10} - p_{11}p_{20}) - (p_{12}^2 - p_{12}p_{22})(p_{13}p_{10} - p_{11}p_{30}).$$

We have demonstrated that the vanishing of the expressions

$$(A)\quad \Delta_{00}^2 p_{ii} + \Delta_{i0}^2 p_{00} + 2\Delta_{i0}\Delta_{00}p_{i0} + \Delta_{00}^2 p_{ij} + \Delta_{i0}\Delta_{j0}p_{00} \qquad (i, j = 1, 2, 3).$$

is a necessary consequence of the vanishing of the system of polynomials (1). Let us now assume that $\Delta_{00} \neq 0$. Then we eliminate p_{11}, p_{12}, p_{13}, p_{22}, p_{23} from the polynomial (3) with the aid of the six expressions (A), and since (3) vanishes, we conclude that

The vanishing of the system of polynomials (1) *carries with it as a necessary consequence the vanishing of the polynomial* (3) *unless* $\Delta_{00} = 0$.

We have demonstrated that if we multiply each of the polynomials of the system (1) by certain multipliers and then sum the results we obtain the expressions (2) and (2′). This is equivalent to solving the set (1) with respect to the quantities p_{ij} where $i, j = 1, 2, \cdots, n$; $i \geqq j$.

Let us consider the effect of interchanging any two columns of the matrix

$$W \equiv \|x_{ij}\| \quad (i = 0, 1, 2, \cdots, n;\ j = 1, 2, \cdots, n).$$

By such a permutation, the system of polynomials (1) is unaltered. The polynomials (2) will be changed, if we operate on them with all possible permutations of the above-described type, into a set of sets of polynomials which we shall denote by the symbols (2_1), (2_2), $\cdots$. This change however is only with regard to outward form. For the above permutations simply enable us to solve the system (1) with respect to another set of coefficients. For example, if we interchange the first and second columns of the matrix W we shall solve (1) with respect to the quantities p_{ij} where $i, j = 0, 2, 3, \cdots, n$; $j \geqq i$. It is now evident that (2), (2_1), (2_2), $\cdots$ are numerically equivalent.

On the other hand, the polynomial (3) is changed by the permutations of the type above described into a chain of other polynomials $(n+1)!$ in number. For example, the permutation (2, 4) transforms (3) into the polynomial

$$(p_{14}p_{13} - p_{11}p_{34})(p_{14}p_{10} - p_{11}p_{40}) - (p_{14}^2 - p_{11}p_{44})(p_{13}p_{10} - p_{11}p_{30}).$$

Let us designate this chain of polynomials by the symbol (3′). We can now announce the following theorem:

If $\Delta_{00} \neq 0$, *the vanishing of the system of polynomials* (1) *is a sufficient condition for the vanishing of the polynomial* (3) *and also of the chain of polynomials* (3′).

Consider the case $\Delta_{00} = 0$. Assume for the moment that $\Delta_{n0} \neq 0$ for

$n \neq 0$. Interchange the first and the nth column of the matrix W. As remarked above, such interchange simply replaces the system (1) by itself and transforms the system (2) into a system numerically equivalent. In other words instead of solving (1) with respect to p_{ij} where $i, j = 1, 2, \cdots, n$, we solve (1) with respect to p_{ij} where $i, j = 0, 1, 2, \cdots, n-1$. In general we may assert

If one of the determinants $\Delta_{10}, \Delta_{20}, \cdots, \Delta_{00}$ does not vanish, the system of polynomials (1) *cannot vanish unless all the polynomials of system* (A) *vanish.*

We now consider the polynomial

$$(4) \qquad (p_{12}^2 - p_{11}p_{22})(p_{13}^2 - p_{11}p_{33}) - (p_{12}p_{13} - p_{11}p_{23})^2.$$

If we equate the polynomials (A) to zero we see at once that the polynomial (4) must also be equal to zero.

Operate on this polynomial with the permutations we have previously used and we shall obtain a chain of polynomials which we shall denote by the symbol (4′). Following the line of reasoning employed in the case of the system of polynomials (4) we demonstrate the following theorem:

If any one of the determinants $\Delta_{i0} \neq 0$ ($i = 0, 1, 2, \cdots, n$), then the vanishing of the system of polynomials (1) *carries with it the vanishing of all the polynomials of system* (3′).

We have now derived two sets of conditions (3) and (3′) which must be satisfied if the system of polynomials (1) has a solution which does not cause all the determinants of matrix W to vanish. These conditions are also sufficient as we shall proceed to show.

Write

$$p_{1i}^2 - p_{11}p_{ii} \equiv w_{ii}, \qquad p_{1i}p_{1j} - p_{11}p_{ij} \equiv w_{ij} \equiv w_{ji} \quad (i, j = 0, 1, 2, \cdots, n).$$

When we equate to zero all the minors of the second order in the determinant $|w_{ij}|$, we obtain a set of necessary conditions. As we shall show, these conditions are sufficient, indeed more than sufficient, but they are retained because of certain invariant properties to be considered in a subsequent paper. We shall demonstrate the sufficiency of these conditions for $n = 3$.*

Consider first the equations belonging to set (2′), that is,

$$\Delta_{00}^2 p_{ii} + 2\Delta_{i0}\Delta_{00}p_{i0} + \Delta_{i0}^2 p_{00} = 0 \quad (i = 1, 2, 3).$$

Write $\Delta_{i0}/\Delta \equiv \lambda_{i0}$. Solving the above equations, we get

$$\lambda_{i0} = \frac{-p_{i0} \mp \sqrt{v_{ii}}}{p_{00}}, \qquad v_{ji} \equiv v_{ij} \equiv p_{0i}p_{0j} - p_{ij}p_{00}.$$

* If $n = 3$ we can construct an equation with the w_{ij} for coefficients which, when written in lines, has coefficients whose vanishing gives the necessary and sufficient conditions for our theorem.

Suppose all the radicals chosen with the positive sign and substitute the above values of λ_{i0} in the three equations of the above set. The results are

$$(5) \qquad v_{12} = \sqrt{v_{11}v_{22}}, \qquad v_{13} = \sqrt{v_{11}v_{33}}, \qquad v_{23} = \sqrt{v_{22}v_{33}}.$$

If all the radicals are chosen with the negative sign the results are essentially the same. We see at once that from the conditions (5) follow the conditions

$$v_{23}v_{13} - v_{12}v_{33} = 0, \qquad v_{23}v_{12} - v_{13}v_{22} = 0, \qquad v_{13}v_{12} - v_{11}v_{23} = 0,^*$$

regardless of the choice of the sign of the radicals.

All other conditions must depend on the six written down above. These six can be written in determinant form.

The same reasoning applies when n is arbitrary.

The polynomials (3) and (3′) are not all independent, and they do not constitute a set of invariants under any linear transformation of coördinates although an invariant property is characterized by the vanishing of all of them simultaneously. They belong to the type of functions which Riquier has utilized to form the conditions that systems of partial differential equations may be passive. If I_s denotes a set of such functions they are transformed into a set I'_r connected with the I_s by relations of the type $I'_r = \sum f_{rs}I_s$ in which the determinant $|f_{rs}|$ is a power of the Jacobian of the transformation.†

If we wish to discover under what conditions a quadratic form in any number of variables can be expressed as the sum of two degenerate quadratic forms we write down the two matrices

$$\begin{aligned} W_1 &\equiv \|x_{ij}\|, \quad i = 1, 2, \cdots, n-1, \qquad j = 1, \cdots, n-1, n, \\ W_2 &\equiv \|x_{ij}\|, \quad i = 1, 2, \cdots, n-1, \qquad j = 0, 1, \cdots, n-1. \end{aligned}$$

The expressions $\Delta_{l,\,0}$ are formed from the matrix $\overline{W}$ in the manner we described at the beginning of this article.

The expressions Δ_{tn} are formed from the matrix W in exactly the same manner.‡ Starting from the system of equations

$$P_{rs} = 0 \quad (r = 1, 2, \cdots, n-1; \quad s = 0, 1, 2, \cdots, n-1, n)$$

we can obtain the canonical form $\Delta_{00}^2 p_{ll} + \Delta_{l,\,0}\Delta_{l,\,0}p_{00} + \Delta_{l,\,0}\Delta_{00}p_{l0}$

* These conditions are retained because of certain invariant conditions to be discussed in the next paper.

† See Riquier, "Les systèmes d'équations aux dérivées partielles." Riquier, being chiefly interested in establishing the existence of solutions of the most general type, has, as far as the author knows, said nothing about the invariant properties of his passivity conditions. But we can immediately verify by a few simple examples that they belong to the type of invariant conditions just defined.

‡ Note that $\Delta_{nn} = \Delta_{00}$.

$+ \Delta_{t0}\Delta_{00}p_{l,\,0} + (\Delta_{tn}\Delta_{l0} + \Delta_{t0}\Delta_{ln})p_{n0} + \Delta_{tn}\Delta_{ln}p_{nn} + \Delta_{ln}\Delta_{nn}p_{tn} + \Delta_{tn}\Delta_{nn}p_{ln}$ $(l, t = 1, 2, \cdots, n-1)$.

When the above system is compatible, it is possible to express the quadratic form as the sum of two degenerate quadratics. The compatibility conditions could be calculated in a given numerical case, but such a calculation is exceeding long. However, we can enunciate the following theorem:

Given the polynomials $P_{rs} = 0$ $(s = 1, 2, \cdots, \lambda, \cdots, \omega;\ r = 1, 2, \cdots, \lambda)$. *The canonical form is a function of* $\lambda(\omega - \lambda)$ *determinants of the matrix* $\|x_{rs}\|$. *The system* $P_{rs} = 0$ *involves* $\lambda\omega$ *variables.* Thus the calculation of the resultant is shortened.

We shall now proceed to treat the case of a quartic form by a similar process. The general quartic may be written symbolically as follows:

$$P_{rrrr} \equiv (a_1x_{1r} + a_2x_{2r} + \cdots + a_nx_{nr} + a_0x_{0r})^4$$

where $a_1^4 = a_{1111}$, $a_1^3a_2 = a_{1112}$, etc. Form the first, second and third polars of each of the above expressions with respect to each of the points A_s whose coördinates are $(x_{1s}, x_{2s}, \cdots, x_{0s})$ respectively, s having the values $1, 2, \cdots, n$.*

If we desire to find the conditions under which the n-ary quartic is a product of a linear factor and a cubic factor, it is necessary to write down in parametric form the equations of a hyperplane. Let us therefore write

$$\rho x_{i1} = y_{i1} + \sum_{k=2}^{n} \lambda^{(k-1)} y_{ik} \quad (i = 1, 2, \cdots, n, 0)$$

and substitute these values for the x's in the polynomials P_{1111}. Equating to zero the coefficients of the λ's, we obtain a system of equations similar to $P_{rrrr} = 0$ and the equations derived from it by the polar process, except that we have the y's in place of the x's.

The argument applied by Clebsch to the conic extends itself immediately to the n-ary quartic. If the system of equations derived in the way just described can be solved in such a way that the solution does not cause all the determinants of the matrix

$$\|y_{ij}\| \quad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n, 0)$$

to vanish, the quartic can be decomposed into a linear and a cubic factor. In the following paragraph when we write $P_{i\cdot\alpha,\,j\cdot\beta,\,k\cdot\gamma,\,l\cdot\delta}$ and $a_{i\cdot\alpha,\,j\cdot\beta,\,k\cdot\gamma,\,l\cdot\delta}$ it means that the subscript α occurs i times, the subscript β, j times, etc.

If we repeat the reasoning applied to a form of the second order we can establish the following identity.

* Denote this system by the symbol (1′).

$$\sum \frac{4!}{i!j!k!l!} M_{t\alpha}^i M_{t\beta}^j M_{t\gamma}^k M_{t\delta}^l P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta}$$
$$\equiv \Delta^4 a_{4\cdot t} + 4\Delta^3 \Delta_t a_{3\cdot t,\,0} + 6\Delta^2 \Delta_t^2 a_{2\cdot t,\,2\cdot 0} + 4\Delta\Delta_t^3 a_{t,\,3\cdot 0} + \Delta_t^4 a_{4\cdot 0}$$
$$(\alpha, \beta, \gamma, \delta, t = 1, 2, \cdots, n;\ i + j + k + l = 4).$$

The summation extends over all possible integral values of the letters involved. Similarly we find that

$$\frac{1}{2\cdot 3\cdot 4} \sum_{i=1}^{n} M_{qi} \frac{\partial}{\partial M_{ti}} \cdot \sum_{i=1}^{n} M_{si} \frac{\partial}{\partial M_{ti}}$$
$$\times \sum_{i=1}^{n} M_{ri} \frac{\partial}{\partial M_{ti}} \sum \left[\frac{4!}{i!j!k!l!} M_{t\alpha}^i M_{t\beta}^j M_{t\gamma}^k M_{t\delta}^l P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta} \right]$$

$$\begin{aligned}(6)\qquad &\equiv \Delta^4 a_{tqsr} + \Delta^3 \{\Delta_t a_{qsr0} + \Delta_q a_{tsr0} + \Delta_s a_{tqr0} + \Delta_r a_{tqs0}\} \\ &+ \Delta^2 \{\Delta_s \Delta_r a_{tq,\,2\cdot 0} + \Delta_q \Delta_r a_{ts,\,2\cdot 0} + \Delta_q \Delta_s a_{tr,\,2\cdot 0} + \Delta_t \Delta_r a_{qs,\,2\cdot 0} \\ &+ \Delta_t \Delta_s a_{qr,\,2\cdot 0} + \Delta_t \Delta_q a_{sr,\,2\cdot 0}\} + \Delta \{\Delta_q \Delta_s \Delta_r a_{t,\,3\cdot 0} + \Delta_t \Delta_q \Delta_r a_{s,\,3\cdot 0} \\ &+ \Delta_t \Delta_q \Delta_s a_{r,\,3\cdot 0} + \Delta_t \Delta_s \Delta_r a_{q,\,3\cdot 0}\} + \Delta_t \Delta_q \Delta_r \Delta_s a_{4\cdot 0}.\end{aligned}$$

To compute the value of the expression

$$\frac{1}{3\cdot 4} \sum_{i=1}^{n} M_{si} \frac{\partial}{\partial M_{ti}} \sum_{i=1}^{n} M_{ri} \frac{\partial}{\partial M_{ti}} \sum \frac{4!}{i!j!k!l!} [M_{t\alpha}^i M_{t\beta}^j M_{t\gamma}^k M_{t\delta}^l P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma\, l\cdot\delta}]$$

let $q = r$ in the right-hand member of the equation (6). To compute the value of the expression

$$\frac{1}{4} \sum_{i=1}^{n} M_{ri} \frac{\partial}{\partial M_{ti}} \sum \frac{4!}{i!j!k!l!} M_{t\alpha}^i M_{t\beta}^j M_{t\gamma}^k M_{t\delta}^l P_{i\cdot\alpha,\, j\cdot\beta,\, k\cdot\gamma,\, l\cdot\delta},$$

in addition to setting $q = r$, we must set $s = t$ in the right-hand member of equation (6).

Equate the right-hand members of all the equations just obtained to zero and denote the resulting system of equations by (A′). These represent a set of conditions which must necessarily be satisfied if the quartic form contains a linear factor. (A′) and (1′) are in "correlation multiplicatoire," for, even as such a relation has been shown to exist in the case of the quadric form, it exists in the case in question. Therefore if all the equations (A′) are satisfied and no inconsistency is introduced, we have the conditions, both necessary and sufficient. To find whether all the equations are consistent, we proceed as before, except that we now have to solve a set of equations of the fourth degree

$$\Delta^4 a_{4t} + 4\Delta^3 \Delta_t a_{3\cdot t,\,0} + 6\Delta^2 \Delta_t^2 a_{2\cdot t,\,2\cdot 0} + 4\Delta\Delta_t^3 a_{t,\,3\cdot 0} + \Delta_t^4 a_{4\cdot 0} = 0$$

to obtain the ratio Δ_t/Δ $(t = 1, 2, \cdots, n)$.

Substituting the resulting values of Δ_t/Δ in the remaining equations of the system (A′) we obtain a set of equations involving the a's alone. If these are satisfied, it is possible to find an infinite number of sets of solutions of our equations which do not cause to vanish all the determinants of the matrix

$$\|x_{ij}\| \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n, 0).$$

We have thus reduced the problem of finding the conditions under which a quartic contains a linear factor to the solution of a binary quartic. Had we attacked the problem directly, i.e., had we written

$$(\alpha x)^4 = (\beta x)(\gamma x)^3,$$

we could have obtained the conditions, but their exact form would have varied according to the order of calculation which we should elect to follow. Proceeding according to one particular order, the author obtained a system of equations similar in some respects to (A′), but less simple. For example in the ternary case the equation corresponding to the first equation of set (A′) is

$$81\Delta^4 - 324\Delta^3 a_{1112} + (432a_{1112}^2 - 54a_{1122})\Delta^2 - (192a_{1112}^3 + 12a_{1222})\Delta \\ - (16a_{1112}a_{1222} - 96a_{1112}^2 a_{1122} - a_{2222}) = 0.$$

This equation has been simplified by setting $a_{1111} = 1$.

As the solution of a quartic equation is generally a long expression, it is better to proceed as follows in a numerical case.

Eliminating Δ_{10} from the equation of the fourth degree in Δ_{10}/Δ and the equation which is linear in Δ_{10} and of the third degree in Δ_{20}, we obtain an equation of the fourth degree in Δ_{20}. Combining this with the equation for Δ_{20} in the set (A′), we may find a common root of the two equations by the usual method of the G. C. D. In order that there may be a common root a certain condition must be satisfied.

Thus one by one we obtain the values of Δ_{10}, Δ_{20}, $\cdots$, Δ_{n0}, and when these are substituted in the equations of the set (A′) the required conditions are obtained.

The cubic form may be treated in a similar way. In the case of the ternary cubic we must determine two conditions and in the case of the quaternary seven.

A theorem analogous to the one on page 151 obviously exists for the quartic case.

SYSTEMS OF TWO LINEAR INTEGRAL EQUATIONS WITH TWO PARAMETERS AND SYMMETRIZABLE KERNELS.

BY MARGARET BUCHANAN.

1. **Introduction.**—Since the time of Sturm and Liouville, numerous memoirs have been written concerning the linear self-adjoint differential equation with a single parameter and with boundary conditions. In the equation

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + (\lambda q - r)u = 0 \tag{a}$$

p and its derivative p', q and r are continuous functions of x independent of the parameter λ, $p > 0$, $q > 0$, and r may have either sign in a given interval. Various writers—Sturm,* Liouville,† Bôcher,‡ Hilbert,§ Stekloff,|| Kneser,¶ Mason,** and others—have proved theorems concerning the existence of characteristic numbers λ_i, and the expansion of more or less arbitrarily given functions into series whose terms are the characteristic functions u_i.

The corresponding theory for a linear integral equation with one parameter and a symmetric kernel,

$$\varphi(s) = \lambda \int_a^b K(s, t)\varphi(t)dt,$$

was developed by Hilbert†† and Schmidt.‡‡ By the use of Green's function,

* Sturm, *Journal de Mathématiques*, Vol. 1 (1836), pp. 106–186.

† *Liouville's Journal*, Vol. 2 (1837), p. 16 and p. 418.

‡ Bôcher, "Encyklopädie der mathematischen Wissenschaften," II A, 7a; *Annals of Mathematics*, Ser. 2, Vol. 13 (1911), p. 71; *Comptes Rendus*, Vol. 140 (1905), p. 928.

§ Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," pp. 39–59.

|| Stekloff, *Annales de la Faculté des sciences de Toulouse*, Ser. 2, Vol. 3, pp. 281–313; *Comptes Rendus*, Vol. CL (1 Semester, 1910), pp. 601–603; ibid., id., pp. 452–454; ibid., Vol. CLI (2 Semester, 1910), pp. 800–802.

¶ Kneser, *Mathematische Annalen*, Vol. 58 (1904), p. 81; ibid., Vol. 60, p. 402; ibid., Vol. 63, p. 477.

** Mason, *Mathematische Annalen*, Vol. 58 (1904), p. 532; *Comptes Rendus*, Vol. 140 (1905), p. 1086; *Transactions of the American Mathematical Society*, Vol. 7 (1906), pp. 337–360; ibid., Vol. 13, p. 516.

†† Hilbert, loc. cit., pp. 46, 49.

‡‡ Schmidt, "Zur Theorie der linearen und nichtlinearen Integralgleichungen," *Mathematische Annalen*, Vol. 63 (1906), p. 433.

Hilbert showed an important connection between a self-adjoint differential system and an integral equation with a symmetric kernel.

For the polar integral equation with a single parameter, that is, an equation with a kernel of the form $a(s)K(s, t)$, in which $a(s)$ is a continuous function that changes sign a finite number of times in the interval considered and $K(s, t)$ is symmetric and positive definite, Hilbert* developed a theory analogous to that for the orthogonal, or symmetric case. If q in equation (a) is a continuous function which changes sign a finite number of times in the interval, if $r \geq 0$ and $p > 0$, the equation leads, not to an orthogonal, but to a polar integral equation.† Hilbert's restriction that q vanish only a finite number of times in the interval has been removed by later writers.‡

A theory similar to that for the orthogonal integral equation has been developed for the integral equation with certain unsymmetric kernels of more general type by A. J. Pell,§ Marty,‖ and others.

Systems of two linear self-adjoint differential equations with two parameters have been considered by Hilbert,¶ who gives an existence theorem for the equations

$$(b)\qquad \begin{aligned} &\frac{d\left(p\dfrac{dy}{dx}\right)}{dx} + (\lambda a + \mu b)y = 0 \qquad (x_1 \leq x \leq x_2),\\ &\frac{d\left(\pi\dfrac{d\eta}{d\xi}\right)}{d\xi} - (\lambda\alpha + \mu\beta)\eta = 0 \qquad (\xi_1 \leq \xi \leq \xi_2), \end{aligned}$$

when $p > 0$, $a > 0$, $\pi > 0$, $\alpha > 0$; p, a, b denoting analytic functions of x and π, α, β analytic functions of ξ. He states that if the further condition $\alpha b - a\beta = 0$ is fulfilled only for a finite number of analytic curves, an arbitrary function of x and ξ that satisfies certain conditions of continuity and the boundary conditions may be developed into a series in terms of the products $y_h(x)\eta_h(s)$, where y_h and η_h indicate simultaneous solutions of the given equations, satisfying the boundary conditions. For the case when

* Hilbert, loc. cit., p. 195.

† Hilbert, loc. cit., p. 205.

‡ Fubini, *Annali di Matematica*, Ser. 3, Vol. 17 (1910), p. 111. Marty, *Comptes Rendus*, Vol. CL (1910), p. 515; ibid., id., pp. 603–606. Garbe, *Mathematische Annalen*, Vol. 76 (1915), pp. 517–547.

§ Pell, *Bulletin of the American Mathematical Society*, Vol. 16 (1910), pp. 513–515; *Transactions of the American Mathematical Society*, Vol. 12 (1911), pp. 165–180.

‖ Marty, *Comptes Rendus*, Vol. 150 (1910), p. 515; ibid., id., p. 605; ibid., id., p. 1031; ibid., id., p. 1499.

¶ Hilbert, loc. cit., p. 263. Oscillation theorems for this case have been proved by Yoshikawa, Göttingen Nachrichten, 1910, pp. 586–594; and Richardson, *Transactions of the American Mathematical Society*, Vol. 13, pp. 22–34.

$\alpha b - \bar{a}\beta = 0$ nowhere in the region, every continuous function of x and ξ, with continuous first and second derivatives, is developable in terms of $y_h(x)\eta_h(s)$.

More recently A. J. Pell† has proved existence and expansion theorems for systems of linear equations with two parameters, of the types

$$(c)\qquad \begin{aligned} u_i &= \lambda \sum_{j=1}^{\infty} k_{ij}u_j + \mu \sum_{j=1}^{\infty} l_{ij}u_j, \\ v_k &= \lambda \sum_{l=1}^{\infty} m_{kl}v_l - \mu \sum_{l=1}^{\infty} n_{kl}v_l; \end{aligned}$$

and

$$(d)\qquad \begin{aligned} u(x) &= \lambda \int_a^b K(x, y)u(y)dy + \mu \int_a^b L(x, y)u(y)dy, \\ v(s) &= \lambda \int_c^d M(s, t)v(t)dt - \mu \int_c^d N(s, t)v(t)dt; \end{aligned}$$

where the matrices and kernels are symmetric.

In the present paper the theory indicated in the last paragraph is extended to equations of type (d) with symmetrizable kernels subject to certain conditions which are stated on page 6. Hilbert's results for the differential equations (b) are obtained by considering a special case. In § 2 it is shown that there exist real values of λ and μ for which the equations (d) with symmetrizable kernels, and the adjoint equations, have continuous solutions u_i, v_i and u_i^*, v_i^*, respectively. Some properties of the solutions are developed in § 3, and the coefficients $\{f_i\}$ of the expansion

$$f(x, s) = \sum_{i=1}^{n} f_i u_i(x)v_i(s)$$

are shown to be of finite norm. In § 4 the expansion of an arbitrary function of two variables is considered. An expansion

$$h(x, s) = \sum_{\alpha} w_\alpha^*(x, s) \int\int h(y, t)u_\alpha(y)v_\alpha(t)dydt,$$

in which

$$w_\alpha^*(x, s) = u_\alpha^*(x)\int N(t, s)v_\alpha^*(t)dt + v_\alpha^*(s)\int L(y, x)u_\alpha^*(y)dy,$$

is obtained for a function $h(x, s)$ that can be expressed by

$$\int\int [K^*(x, y)f(y, t)N^*(t, s) + L^*(x, y)f(y, t)M^*(t, s)]dydt$$

where $f(y, t)$ is any continuous function of y and t, and K^*, L^*, M^*, N^* are symmetric kernels connected with K, L, M, and N, respectively.

† Pell, "Linear Equations with Two Parameters," *Transactions of the American Mathematical Society*, Vol. 23, No. 2 (1922), pp. 198–211.

That functions satisfying less stringent conditions may be expanded in terms of other characteristic functions connected with the integral equations (d) is shown in § 5 for the two important cases where the kernels are symmetrizable functions of well-known types:

$$\text{I.}\quad K(x, y) = \int K(x, \xi)T(\xi, y)d\xi, \qquad L(x, y) = \int L(x, \xi)T(\xi, y)d\xi,$$

$$M(s, t) = \int M(s, \eta)\overline{T}(\eta, t)d\eta, \qquad N(s, t) = \int N(s, \eta)\overline{T}(\eta, t)d\eta.$$

$$\text{II.}\quad K(x, y) = a(x)T(x, y), \qquad L(x, y) = b(x)T(x, y),$$

$$M(s, t) = \bar{a}(s)\overline{T}(s, t), \qquad N(s, t) = \bar{b}(s)\overline{T}(s, t).$$

Kernels of this type have already been mentioned in connection with the polar integral equation. Differential equations (b) lead to integral equations with kernels of the second type.

In obtaining the results as to the existence of solutions and the expansion of arbitrary functions, the method used is the reduction of equations (d) with symmetrizable kernels to equations (c) with symmetric matrices and the application of the theory previously developed for equations in infinitely many variables with two parameters.

2. **Existence of Solutions.**—Consider the system of linear integral equations

$$\begin{aligned} u(x) &= \lambda\int_a^b K(x, y)u(y)dy + \mu\int_a^b L(x, y)u(y)dy, \\ v(s) &= \lambda\int_c^d M(s, t)v(t)dt - \mu\int_c^d N(s, t)v(t)dt, \end{aligned} \tag{1}$$

and the adjoint system

$$\begin{aligned} u^*(x) &= \lambda\int_a^b u^*(y)K(y, x)dy + \mu\int_a^b u^*(y)L(y, x)dy, \\ v^*(s) &= \lambda\int_c^d v^*(t)M(t, s)dt - \mu\int_c^d v^*(t)N(t, s)dt. \end{aligned} \tag{2}$$

In (1) and (2) the kernels K and L are real continuous functions of the real variables x and y in $a \leq x \leq b$, $a \leq y \leq b$, M and N are real continuous functions of the real variables s and t in $c \leq s \leq d$, $c \leq t \leq d$; L and N are positive definite* or have only positive characteristic numbers and $KN + LM \not\equiv 0$. Further, there exist continuous and symmetric positive definite kernels T and $\overline{T}$, such that K and L are symmetrizable on the left by T, M and N by $\overline{T}$, that is, $\int_a^b T(x, \xi)K(\xi, y)d\xi$, $\int_a^b T(x, \xi)L(\xi, y)d\xi$,

* The kernel L is positive definite if $\int\int f(x)L(x, y)f(y) > 0$, for functions f, not null functions, integrable and with squares integrable in the sense of Lebesgue, except on a set of points of measure zero.

$\int_c^d \overline{T}(s,\ \eta)M(\eta,\ t)d\eta, \int_c^d \overline{T}(s,\ \eta)N(\eta,\ t)d\eta$ are symmetric functions. We shall use the following notation:

$$\int_a^b T(x,\ \xi)K(\xi,\ y)d\xi = \int TK = K^*, \qquad \int TL = L^*,$$

$$\int_c^d \overline{T}(s,\ \eta)M(\eta,\ t)d\eta = \int \overline{T}M = M^*, \qquad \int \overline{T}N = N^*.$$

In general,

$$\textstyle\int AB = \int A(x,\ z)B(z,\ y)dz, \qquad \int fA = \int f(y)A(y,\ x)dy,$$
$$\textstyle\int Af = \int A(x,\ y)f(y)dy.^*$$

To show that there exist real values of λ and μ for which systems (1) and (2) have continuous solutions, system (1) is reduced to a system of linear equations in infinitely many unknowns, with symmetric matrices. In the process of reduction, use is made of the expansion†

$$\int fg = \sum_{i=1}^{\infty} \int f\psi_i \int g\varphi_i \tag{3}$$

where f and g are continuous functions, g is of the form $\int Th$, and $\{\varphi_i,\ \psi_i\}$ is a closed biorthogonal system of continuous functions such that $\psi_i = \int T\varphi_i$. Let $\{\overline{\varphi}_k,\ \overline{\psi}_k\}$ be a second closed biorthogonal system of continuous functions such that $\overline{\psi}_k = \int \overline{T}\overline{\varphi}_k$. For $\{\varphi_i,\ \psi_i\}$ and $\{\overline{\varphi}_k,\ \overline{\psi}_k\}$ it is convenient to use functions related to the characteristic functions χ_i and $\overline{\chi}_k$ of the symmetric positive definite kernels T and $\overline{T}$, respectively. Since the characteristic numbers of T and $\overline{T}$ are positive,‡ we indicate them by a_i^2 and b_k^2 and write

$$\chi_i(x) = a_i^2 \textstyle\int T(x,\ y)\chi_i(y)dy, \qquad \overline{\chi}_k(s) = b_k^2 \textstyle\int \overline{T}(s,\ t)\overline{\chi}_k(t)dt,$$

which become

$$\frac{\chi_i(x)}{a_i} = \int T(x,\ y)a_i\chi_i(y)dy, \qquad \frac{\overline{\chi}_k(s)}{b_k} = \int \overline{T}(s,\ t)b_k\overline{\chi}_k(t)dt. \tag{4}$$

Let

$$\varphi_i = a_i\chi_i, \qquad \psi_i = \frac{\chi_i}{a_i}, \qquad \overline{\varphi}_k = b_k\overline{\chi}_k, \qquad \overline{\psi}_k = \frac{\overline{\chi}_k}{b_k},$$

* As we shall consider only the square $a \leq x \leq b$, $a \leq y \leq b$, for the variables x and y, and the square $c \leq s \leq d$, $c \leq t \leq d$, for the variables s and t, the limits of integration and statements as to the range of the variables will be omitted. Wherever there is no ambiguity, we shall omit the variables.

† Pell, "Biorthogonal Systems of Functions," *Transactions of the American Mathematical Society*, Vol. XII, p. 147.

‡ Lalesco, "Introduction à la Théorie des Équations Intégrales," p. 71.

then since we may assume that the systems $\{\chi_i\}$, $\{\overline{\chi}_k\}$ are orthogonal and normalized, we have

$$\int \varphi_i \psi_j = \delta_{ij} \quad \text{and} \quad \int \overline{\varphi}_k \overline{\psi}_l = \delta_{kl}, \quad \text{where} \quad \delta_{\xi\eta} = \begin{cases} 1 & \xi = \eta, \\ 0 & \xi \neq \eta. \end{cases}$$

The equations (4) give the relations

$$\psi_i(x) = \int T(x, y)\varphi_i(y)dy, \quad \overline{\psi}_k(s) = \int \overline{T}(s, t)\overline{\varphi}_k(t)dt.$$

We now proceed to reduce the system (1) to a system of linear equations in infinitely many unknowns. Multiplying the first equation of (1) by $\psi_i(x)$, integrating, and then expanding the second member by (3), we have

$$\int u(x)\psi_i(x)dx = \sum_{j=1}^{\infty} \int\int \varphi_i(\xi) \int T(\xi, x)[\lambda K(x, y) + \mu L(x, y)]dx \cdot \varphi_j(y)d\xi dy \cdot \int u(y)\psi_j(y)dy$$

or

$$\int u\psi_i = \sum_{j=1}^{\infty} \int\int \varphi_i \int T(\lambda K + \mu L)\varphi_j \cdot \int u\psi_j, \tag{5}$$

and similarly from the second equation of (1) we have

$$\int v\overline{\psi}_k = \sum_{l=1}^{\infty} \int\int \overline{\varphi}_k \int \overline{T}(\lambda M - \mu N)\overline{\varphi}_l \cdot \int v\overline{\psi}_l. \tag{6}$$

Equations (5) and (6) may be written in the form

$$\begin{aligned} x_i &= \lambda \sum_{j=1}^{\infty} k_{ij}x_j + \mu \sum_{j=1}^{\infty} l_{ij}x_j, \\ y_k &= \lambda \sum_{l=1}^{\infty} m_{kl}y_l - \mu \sum_{l=1}^{\infty} n_{kl}y_l, \end{aligned} \tag{7}$$

where

$$x_i = \int u\psi_i, \qquad k_{ij} = \int\int \varphi_i K^* \varphi_j, \qquad l_{ij} = \int\int \varphi_i L^* \varphi_j,$$
$$y_k = \int v\overline{\psi}_k, \qquad m_{kl} = \int\int \overline{\varphi}_k M^* \overline{\varphi}_l, \qquad n_{kl} = \int\int \overline{\varphi}_k N^* \overline{\varphi}_l.$$

The matrices K, L, M, N of (7) are symmetric, since the kernels K^*, L^*, M^*, N^* are symmetric, and the sum of the squares of the elements of each is convergent. For example, the convergency of $\sum_{i,j} k_{ij}^2$ is evident from the following

$$k_{ij}^2 = \int\int \varphi_i(x) \int T(x, \xi)K(\xi, y)d\xi\varphi_j(y)dxdy \times \int\int \varphi_i(x) \int T(y, \xi)K(\xi, y)d\xi \cdot \varphi_j(y)dxdy.$$

$$= a_i \int\int \chi_i(x)K(\xi, x)\frac{\chi_j(\xi)}{a_j}\,dxd\xi \cdot a_j \int\int \chi_j(x)K(\xi, x)\frac{\chi_i(\xi)}{a_i}\,dxd\xi.$$

But

$$\int\int \chi_i K\chi_j \cdot \int\int \chi_j K\chi_i \leq [\int\int \chi_i K\chi_j]^2 + [\int\int \chi_j K\chi_i]^2,$$

and since†

$$\sum_{ij}\left[\int\int \chi_i(\xi)K(\xi, x)\chi_j(x)d\xi dx\right]^2 \leq \int\int [K(\xi, x)]^2 d\xi dx,$$

the series $\sum_{ij} k_{ij}^2$ is convergent.

The two matrices l_{ij} and n_{kl} are of positive definite type, if the kernels L^* and N^* are themselves positive definite. In proving L^* and N^* positive definite, use is made of the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\bar{\varphi}_k, \bar{\psi}_k\}$ defined by the relations‡

$$\begin{aligned} &\varphi_i = \alpha_i^2 \int L\varphi_i, && \psi_i = \alpha_i^2 \int \psi_i L, && \psi_i = \int T\varphi_i, \\ &\bar{\varphi}_k = \beta_k^2 \int N\bar{\varphi}_k, && \bar{\psi}_k = \beta_k^2 \int \bar{\psi}_k N, && \bar{\psi}_k = \int \bar{T}\bar{\varphi}_k. \end{aligned} \tag{8}$$

By the expansion (3)

$$\begin{aligned} &\int\int f(x)L^*(x, y)f(y)dxdy \\ &\qquad = \sum_i \int\int T(x, \xi)f(x)dx \cdot \varphi_i(\xi)d\xi \cdot \int\int L(\xi, y)f(y)dy \cdot \psi_i(\xi)d\xi \\ &\qquad = \sum_i \frac{[\int \psi_i(x)f(x)dx]^2}{\alpha_i^2}. \end{aligned} \tag{9}$$

Similarly,

$$\int\int g(s)N^*(s, t)g(t)dsdt = \sum_k \frac{[\int \bar{\psi}_k(s)g(s)ds]^2}{\beta_k^2}.$$

Therefore the kernels L^* and N^*, and consequently, since matrices obtained from the kernels by another set of biorthogonal functions corresponding to T and $\bar{T}$ would differ from K, L, M, N only by orthogonal matrices, the matrices l_{ij} and n_{kl} are positive definite.

Inasmuch as the matrices K, L, M, and N of (7) are symmetric and such that the sum of the squares of the elements of each one is convergent, inasmuch as L and N are positive definite and $k_{ij}n_{kl} + l_{ij}m_{kl} \not\equiv 0$, there exist real values of λ and μ for which the system (7) has solutions x_i, y_k of

† Hilbert, loc. cit., p. 181.

‡ Pell, "Existence Theorems for Certain Unsymmetric Kernels," *Bulletin of the American Mathematical Society*, Vol. 16, p. 515. Marty, *Comptes Rendus*, February 28 and April 25, 1910.

finite norm,† and to any pair of parameter values λ, μ there corresponds only a finite number of linearly independent solutions x_i, y_k.

We now pass from the equations (7) in infinitely many variables to the integral equations (2). By means of x_i, y_k, solutions of finite norm of (7) corresponding to a definite set of parameter values denoted by λ_0, μ_0, we obtain a pair of continuous solutions of (2) corresponding to the same parameter values λ_0, μ_0. Consider, in particular, the first equation of (7)

$$x_i = \sum_j \int\int \varphi_i(\xi)[\lambda_0 K^*(\xi, z) + \mu_0 L^*(\xi, z)]\varphi_j(z) d\xi dz \cdot x_j;$$

multiply both members by $\int [\lambda_0 K(y, x) + \mu_0 L(y, x)]\psi_i(y)dy$, and sum with respect to i. The result is

$$\sum_{i=1} \int [\lambda_0 K(y, x) + \mu_0 L(y, x)]\psi_i(y)dy \cdot x_i$$

$$= \sum_{ij} \int [\lambda_0 K(y, x) + \mu_0 L(y, x)]\psi_i(y)dy$$

$$\times \int\int \varphi_i(\xi)[\lambda_0 K^*(\xi, z) + \mu_0 L^*(\xi, z)]\varphi_j(z) d\xi dz \cdot x_j,$$

which reduces to

$$\sum_{i=1} \int [\lambda_0 K(w, x) + \mu_0 L(w, x)]\psi_i(w)dw \cdot x_i$$

$$= \int [\lambda_0 K(y, x) + \mu_0 L(y, x)] \tag{10}$$

$$\times \sum_j \int [\lambda_0 K(w, y) + \mu_0 L(w, y)]\psi_j(w) dw dy \cdot x_j.$$

From (10) it is evident that for the parameter values or characteristic numbers λ_0 and μ_0, the continuous function

$$u^*(x) = \sum_j \int [\lambda_0 K(w, x) + \mu_0 L(w, x)]\psi_j(w)dw \cdot x_j$$

is a solution of

$$u^*(x) = \lambda_0 \int u^*(y) K(y, x) dy + \mu_0 \int u^*(y) L(y, x) dy, \tag{2)'$$

which is the first equation of the adjoint system (2). Therefore by the Fredholm theory,

$$u(x) = \lambda_0 \int K(x, y) u(y) dy + \mu_0 \int L(x, y) u(y) dy \tag{1)'$$

† Pell, *Transactions of the American Mathematical Society*, Vol. 23, No. 2, p. 203.

has a continuous solution $u(x)$, and by the same theory, if n denotes the number of linearly independent solutions $u_1^*, u_2^*, \cdots, u_n^*$ of (2)′, the equation (1)′ has exactly n linearly independent solutions $u_1, u_2, \cdots, u_n$. In the same way it may be shown from the second equation of (7) that for λ_0, μ_0 the second equation of the adjoint system (2) has a continuous solution $v^*(s)$. It follows that

$$v(s) = \lambda_0 \int M(s, t)v(t)dt - \mu_0 \int N(s, t)v(t)dt$$

has a continuous solution $v(s)$, and that, corresponding to a pair of characteristic numbers λ_0, μ_0, the number of linearly independent solutions v_k is the same as the number of linearly independent solutions v_k^*.

That a simple relation exists between the solutions of the systems (1) and (2) appears from the following: the equations (1) when multiplied by $T(x, \xi)$ and $\overline{T}(s, \eta)$, respectively, and integrated, become

$$\int T(x, \xi)u(x)dx = \lambda \int\int K(x, \xi)\,T(x, y)u(y)dydx + \mu \int\int L(x, \xi)\,T(x, y)u(y)dydx,$$

$$\int T(s, \eta)v(s)ds = \lambda \int\int M(s, \eta)\,\overline{T}(s, t)v(t)dtds - \mu \int\int N(s, \eta)\overline{T}(s, t)v(t)dtds,$$

and these equations show that $\int Tu$ and $\int \overline{T}v$ are solutions of the adjoint system (2). On account of the (1, 1) correspondence already noted between the solutions u and u^*, and between v and v^*, we may say

$$u_i^* = \int Tu_i, \qquad v_k^* = \int \overline{T}v_k.$$

These results give the following theorem:

THEOREM 1: *If $K(x, y)$, $L(x, y)$, $M(s, t)$, $N(s, t)$ are real continuous functions, if L and N are positive definite or have only positive characteristic numbers, and $KN + LM \neq 0$, and if there exist continuous symmetric positive definite kernels T and $\overline{T}$ such that $\int TK$, $\int TL$, $\int \overline{T}M$, $\int \overline{T}N$ are symmetric, there exist values λ and μ, necessarily real, for which the system of equations* (1) *has continuous solutions u, v, not identically zero, and for which the adjoint system* (2) *has continuous solutions u^*, v^*, not identically zero. The solutions of systems* (1) *and* (2) *are connected by the relations $u_i^* = \int Tu_i$, $v_k^* = \int \overline{T}v_k$.*

3. **Properties of Solutions.**—Let $u_i(x)$, $v_i(s)$ be solutions of system (1) corresponding to λ_i, μ_i and $u_k(x)$, $v_k(s)$ solutions corresponding to λ_k, μ_k. Then from the equations for u_i and u_k, by using as multipliers $u_k^*(x)$ and $u_i^*(x)$ and integrating, we obtain

$$\int\int u_i T u_k = \lambda_i \int\int\int u_k TKu_i + \mu_i \int\int\int u_k TLu_i,$$

$$\int\int u_k T u_i = \lambda_k \int\int\int u_i TKu_k + \mu_k \int\int\int u_i TLu_k,$$

and therefore, since the kernels $\int TK$ and $\int TL$ are symmetric,

(11) $$(\lambda_i - \lambda_k)\int\int\int u_k TKu_i + (\mu_i - \mu_k)\int\int\int u_k TLu_i = 0.$$

In the same way, from the equations in v_i and v_k, there results

(12) $$(\lambda_i - \lambda_k)\int\int\int v_k \overline{T}Mv_i - (\mu_i - \mu_k)\int\int\int v_k \overline{T}Nv_i = 0.$$

If $\lambda_i \neq \lambda_k$ and $\mu_i \neq \mu_k$, the determinant of the coefficients of (11) and (12) must vanish, that is,

(13) $$\begin{vmatrix} \int\int u_k^* Ku_i & \int\int u_k^* Lu_i \\ \int\int v_k^* Mv_i & -\int\int v_k^* Nv_i \end{vmatrix} = 0.$$

The equations

$$\int\int u_i Tu_k = \lambda_i \int\int u_k TKu_i + \mu_i \int\int u_k TLu_i,$$
$$\int\int v_i \overline{T}v_k = \lambda_i \int\int v_k \overline{T}Mv_i - \mu_i \int\int v_k \overline{T}Nv_i,$$

give, on the elimination of μ and the use of the relation (13),

$$\begin{vmatrix} \int\int u_k^* Lu_i & \int u_k^* u_i \\ -\int\int v_k^* Nv_i & \int v_k^* v_i \end{vmatrix} = 0.$$

Hence the matrix

$$\begin{pmatrix} \int\int u_k^* Ku_i & \int\int u_k^* Lu_i & \int u_k^* u_i \\ \int\int v_k^* Mv_i & -\int\int v_k^* Nv_i & \int v_k^* v_i \end{pmatrix}$$

is of rank < 2, if $i \neq k$.

To show that $\int u_i u_i^* \int\int v_i^* Nv_i + \int v_i v_i^* \int\int u_i^* Lu_i \neq 0$, we have only to write the expression in the form

$$\int\int u_i Tu_i \int\int v_i N^* v_i + \int\int v_i \overline{T}v_i \int\int u_i L^* u_i,$$

where each term > 0 because of the positive definite character of T, $\overline{T}$, L^*, and N^*. Since $\int u_i u_i^* \int\int v_i^* Nv_i + \int v_i v_i^* \int\int u_i^* Lu_i > 0$ we may assume that the solutions u_i and v_i have been multiplied by such constants as to make

$$\int u_i u_i^* \int\int v_i^* Nv_i + \int v_i v_i^* \int\int u_i^* Lu_i = 1.$$

We have, therefore, for (u_i, v_i), (u_k, v_k), corresponding to different characteristic numbers, the relation

(14) $$\int\int u_i u_k^* \int\int v_k^* Nv_i + \int v_i v_k^* \int\int u_k^* Lu_i = \int\int u_i(x) v_i(s) w_k^*(x, s)\,dx\,ds = \delta_{ik}$$

where

(15) $$w_\alpha^*(x, s) = u_k^*(x)\int N(t, s) v_k^*(t)\,dt + v_k^*(s)\int L(y, x) u_k^*(y)\,dy.$$

From every system of n linearly independent solutions of the equations (1) corresponding to a particular pair of characteristic numbers it is possible to build up by linear combinations n linearly independent functions that satisfy the relation (14). Let $u_1, u_2, \cdots, u_n$; $v_1, v_2, \cdots, v_n$ be solutions of (1) corresponding to λ_0, μ_0. We assume, as above, that $\int\int u_1 v_1 w_1^* = 1$, and then determine c_i and d_i so that

$$\begin{aligned} &\int\int (u_i - c_i u_1)(v_i - d_i v_1) w_1^* \\ (16) \qquad &\qquad = \int\int u_i v_i w_1^* - c_i \int\int u_1 v_i w_1^* + d_i(c_i - \int\int u_i v_1 w_1^*) = 0, \\ &\qquad\qquad i = 2, 3, \cdots, n. \end{aligned}$$

If $\int\int u_i v_i w_1^* = 0$, take $c_i = d_i = 0$, and the above equation is satisfied. If $\int\int u_i v_i w_1^* \neq 0$, choose $c_i \neq \int\int u_i v_1 w_1^*$ and from (16) determine d_i. Let $u_1 = \bar{u}_1, u_i - c_i u_1 = \bar{u}_i$; $v_1 = \bar{v}_1, v_i - d_i v_1 = v_i$; $i = 2, 3, \cdots, n$. Then the new system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$; $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n$ is such that $\int\int \bar{u}_i \bar{v}_i \bar{w}_1^* = \delta_{i1}$. Proceeding as before, we determine $\bar{c}_i$ and $\bar{d}_i$ to satisfy the equation $\int\int (\bar{u}_i - \bar{c}_i \bar{u}_2)(\bar{v}_i - \bar{d}_i \bar{v}_2)\bar{w}_2^* = 0$, $i = 3, 4, \cdots, n$. By repeating the process indicated above, we obtain finally a system $U_1, U_2, \cdots, U_n$; $V_1, V_2, \cdots, V_n$, where U_i and V_i are linear homogeneous functions with constant coefficients of $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, respectively, with the constants so determined that $\int\int U_i V_i W_k^* = \delta_{ik}$.

If the series

$$f(x, s) = \sum_{i=1}^{\infty} f_i u_i(x) v_i(s) \tag{17}$$

is uniformly convergent, the coefficients of $u_i v_i$ may be expressed in a form analogous to the Fourier coefficients in the expansion

$$g(s) = \sum_{\nu} \varphi_\nu(s) \int_a^b g(t)\psi_\nu(t)dt,$$

where $\{\varphi_\nu, \psi_\nu\}$ is a biorthogonal system. Multiply (17) by $u_k^*(x)\int N(t, s)v_k^*(t)dt + v_k^*(s)\int L(y, x)u_k^*(y)dy$, integrate, and use the relation (14), thus obtaining

$$\begin{aligned} f_k &= \int\int\int u_k^*(x) f(x, s) N(t, s) v_k^*(t) dx ds dt + v_k^*(s) f(x, s) L(y, x) u_k^*(y) ds dx dy \\ &= \int\int f(x, s) w_k^*(x, s) dx ds. \end{aligned}$$

If the series

$$f(x, s) = \sum_{i=1}^{\infty} A_i w_i^*(x, s)$$

is uniformly convergent, the coefficients A_i may be expressed in terms of the solutions u_i and v_i. Multiplying the equation by $u_k(x)v_k(s)$ and

integrating, we have

$$\int\int f(x, s)u_k(x)v_k(s)dxds$$

$$= \sum_{i=1}^{\infty} A_i \left[\int v_k v_i^* \int\int u_i^* L u_k + \int u_k u_i^* \int\int v_i^* N v_k \right].$$

Hence

$$A_i = \int\int f(x, s)u_i(x)v_i(s)dxds.$$

Again, if a function $f(x, s)$ be expressed by a uniformly convergent series of the form

$$f(x, s) = \sum_i B_i u_i^*(x)v_i^*(s),$$

or

$$f(x, s) = \sum_i C_i w_i(x, s),$$

where

(18) $$w_i(x, s) = u_i(x)\int N(s, t)v_i(t)dt + v_i(s)\int L(x, y)v_i(y)dy,$$

the coefficients are found to be

$$B_i = \int\int f(x, s)w_i(x, s)dxds,$$

$$C_i = \int\int f(x, s)u_i^*(x)v_i^*(s)dxds.$$

Let $f(x, s)$ and $g(x, s)$ be continuous functions and suppose that the series on the right of (17) is uniformly convergent. After multiplication of (17) by $T(x, z)\overline{T}(s, \eta)[\int g(z, \xi)N(\eta, \xi)d\xi + \int g(\xi, \eta)L(z, \xi)d\xi]$, and integration, we have

$$\int\int\int\int f(x, s)T(x, z)N^*(s, \xi)g(z, \xi)dxdsd\xi dz$$

$$+ \int\int\int\int f(x, s)\overline{T}(s, \eta)L^*(x, \xi)g(\xi, \eta)dxdsd\eta d\xi$$

$$= \sum_i f_i \left[\int\int\int u_i^*(z)g(z, \xi)N(\eta, \xi)v_i^*(\eta)dzd\xi d\eta \right.$$

$$\left. + \int\int\int v_i^*(\eta)g(\xi, \eta)L(z, \xi)u_i^*(z)d\eta d\xi dz \right]$$

$$= \sum_i f_i g_i.$$

The coefficients $f_i = \int f w_i^*$ of a continuous function $f(x, s)$ are of finite norm. In order to prove that $\sum_i f_i^2$ is convergent, we establish, for $F(x, s)$

a continuous function, the relation

$$(19)\quad \int\int\int\int[F(x, s)L^*(x, y)F(y, t)\overline{T}(s, t) + F(x, s)N^*(s, t)F(y, t)T(x, y)]dxdsdtdy \geq 0.$$

This inequality is a direct result of the positive definite character of L^* and N^*, and is evident if the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$ defined by (8) are used in the expansion of the expression on the left of (19). By the expansion (3) and by substitutions from (8),

$$\begin{aligned}\int\int\int\int &[F(x, s)L^*(x, y)F(y, t)\overline{T}(s, t) \\ &\qquad + F(x, s)N^*(s, t)F(y, t)T(x, y)]dxdsdtdy \\ &= \sum_k \int\int\int F(x, s)\overline{\psi}_k(s)ds \cdot L^*(x, y)\int\int \overline{T}(s, t)F(y, t)dt \cdot \overline{\varphi}_k(s)dsdydx \\ &\quad + \sum_i \int\int\int F(x, s)\psi_i(x)dxN^*(s, t)\int\int T(x, y)F(y, t)dy \cdot \varphi_i(x)dxdtds \\ &= \sum_k \int\int\int F(x, s)\overline{\psi}_k(s)dsL^*(x, y)\int F(y, t)\overline{\psi}_k(t)dtdydx \\ &\quad + \sum \int\int\int F(x, s)\psi_i(x)dx \cdot N^*(s, t)\int F(y, t)\psi_i(y)dydtds,\end{aligned}$$

where each term is positive because L^* and N^* are positive definite Hence the relation (19) is established. To show $\sum_i f_i^2$ convergent, substitute $f(x, s) - \sum_i f_i u_i(x)v_i(s)$ for $F(x, s)$ in (19), thus obtaining

$$\begin{aligned}\int\int\int\int &[f(x, s)L^*(x, y)f(y, t)\overline{T}(s, t) + f(x, s)N^*(s, t)f(y, t)T(x, y)]dxdydsdt \\ &- \sum_i f_i \int\int\int\int [v_i^*(s)f(x, s)L(\xi, x)u_i^*(\xi)dsdxd\xi \\ &\qquad + u_i^*(x)f(x, s)N(\eta, s)v_i^*(\eta)dxdsd\eta] \\ &- \sum_i f_i \int\int\int\int [v_i^*(t)f(y, t)L(\xi, y)u_i^*(\xi)dtdyd\xi \\ &\qquad + u_i^*(y)f(y, t)N(\eta, t)v_i^*(\eta)dydtd\eta] \\ &+ \sum_i f_i^2 \left[\int u_i(x)u_i^*(x)dx\int\int v_i^*(\eta)N(\eta, s)v_i(s)d\eta ds \right.\\ &\qquad \left. + \int v_i(s)v_i^*(s)ds\int\int u_i^*(\xi)L(\xi, x)u_i(x)d\xi dx\right] \geq 0.\end{aligned}$$

This reduces to

$$\int\int\int\int[fL^*f\overline{T}+fN^*fT]-2\sum_i f_i^2+\sum_i f_i^2\geq 0,$$

which gives

$$\sum_i f_i^2\leq\int\int\int\int[fL^*f\overline{T}+fN^*fT]. \tag{20}$$

We conclude from (20) that the sequence of coefficients $\{f_i\}$ is of finite norm.

4. **Expansion of Arbitrary Functions of Two Variables.**—For a system of two linear integral equations with symmetric kernels, it has been shown† that when f and g are continuous functions

$$\int\int\int\int[K(x,y)f(x,s)N(s,t)g(y,t)$$
$$+L(x,y)f(x,s)M(s,t)g(y,t)]dxdydsdt=\sum_\alpha\frac{f_\alpha g_\alpha}{\lambda_\alpha},$$

where $f_\alpha=\int\int f(x,s)w_\alpha(x,s)dxds$, and that

$$\int\int K(x,y)f(y,t)N(s,t)+L(x,y)f(y,t)M(s,t)dydt=\sum_\alpha\frac{w_\alpha(x,s)f_\alpha}{\lambda_\alpha}.$$

Corresponding forms may be obtained for the symmetrizable case as a result of the expansion† of the determinant matrix

$$k_{ij}n_{kl}+l_{ij}m_{kl}=\sum_\alpha\frac{w_{\alpha ik}w_{\alpha jl}}{\lambda_\alpha}, \tag{21}$$

where $w_{\alpha ik}=y_{\alpha k}\sum_j l_{ij}x_{\alpha j}+x_{\alpha i}\sum_l n_{kl}y_{\alpha l}$, $x_{\alpha i}$ and $y_{\alpha k}$ denoting solutions of (7) corresponding to the characteristic numbers λ_α, μ_α. Since

$$w_{\alpha ik}=\int v_\alpha\overline{\psi}_k\sum_j\int\int\varphi_iL^*\varphi_j\int u_\alpha\psi_j+\int u_\alpha\psi_i\sum_l\int\int\varphi_kN^*\varphi_l\int v_\alpha\overline{\psi}_l,$$

by means of (21), the following equality may be verified:

$$\int\int\int\int K^*(x,y)f(x,s)N^*(s,t)g(y,t)$$
$$+L^*(x,y)f(x,s)M^*(s,t)g(y,t)]dxdsdydt=\sum_\alpha\frac{f_\alpha g_\alpha}{\lambda_\alpha}. \tag{22}$$

As $f(x,s)$ and $g(x,s)$ are any two continuous functions, it follows from (22) that

$$K^*(x,y)N^*(s,t)+L^*(x,y)M^*(s,t)=\sum_\alpha\frac{w_\alpha^*(x,s)w_\alpha^*(y,t)}{\lambda_\alpha},$$

if the series on the right is uniformly convergent.

† Pell, *Transactions of the American Mathematical Society*, Vol. 23, No. 2, p. 208.

The problem of the expansion of arbitrary functions in terms of the solutions u_α and v_α, u_α^* and v_α^*, and in terms of w_α and w_α^* will now be considered. On account of (21), we have the relation

$$\int\int [K^*(x, y)f(y, t)N^*(s, t) + L^*(x, y)f(y, t)M^*(s, t)]dydt = \sum_\alpha \frac{w_\alpha^*(x, s)f_\alpha}{\lambda_\alpha}, \tag{23}$$

if the series on the right is uniformly convergent. As $\{f_\alpha\}$ is known to be of finite norm (§ 3), the series $\sum_\alpha w_\alpha^*(x, s)f_\alpha/\lambda_\alpha$ is absolutely uniformly convergent if $\{w_\alpha^*(x, s)/\lambda_\alpha\}$ is of finite norm. In proving the convergence of

$$\sum_\alpha \left(\frac{w_\alpha^*(x, s)}{\lambda_\alpha}\right)^2$$

we employ again the two biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$, corresponding to the unsymmetric kernels L and N, respectively. If there exists a continuous function $A(x, y, s, t)$, such that

$$\int\int\int\int \varphi_i(x)\psi_j(y)A(x, y, s, t)\overline{\varphi}_k(s)\overline{\psi}_l(t)dxdydsdt = a_{ikjl}\frac{\alpha_j^2\beta_l^2}{\alpha_j^2 + \beta_l^2}, \tag{24}$$

where

$$\begin{aligned} a_{ikjl} &= k_{ij}^* \cdot \frac{\delta_{kl}}{\beta_k^2} + m_{kl}^* \cdot \frac{\delta_{ij}}{\alpha_i^2} \\ &= \int\int \psi_i(y)K(y, x)\varphi_j(x)dydx\,\frac{\delta_{kl}}{\beta_k^2} + \int\int \overline{\psi}_k(t)M(t, s)\overline{\varphi}_l(s)dtds \cdot \frac{\delta_{ij}}{\alpha_i^2}, \end{aligned}$$

it follows immediately that $\{w_\alpha^*(x, s)/\lambda_\alpha\}$ is of finite norm, for

$$\frac{w_\alpha^*(x, s)}{\lambda_\alpha} = \int\int A(x, y, s, t)w_\alpha^*(y, t)dydt, \tag{25}$$

and since $\int\int f w_i^* = f_i$, $\{\int\int A(x, y, s, t)w_\alpha^*(y, t)dydt\}$ is of finite norm. The relation expressed by (25) may be verified by substituting in (25) the value of w_α^* given by (15), expanding the second member in terms of the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\varphi_k, \overline{\psi}_k\}$, and finally multiplying the equation by $\varphi_i(x)\overline{\varphi}_k(s)$ and integrating with respect to x and s. The result of these operations is an equality already established †

$$c_{ik}^2 x_{ik}^* = \lambda_\alpha \sum_{jl} a_{ikjl}x_{jl}^* \tag{26}$$

in which $x_{ik}^* = \int u_\alpha^*\varphi_i \int v_\alpha^*\varphi_k$, $c_{ik}^2 = \dfrac{1}{\alpha_i^2} + \dfrac{1}{\beta_k^2}$, and a_{ikjl} is the matrix already defined.

† Pell, *Transactions of the American Mathematical Society*, Vol. 23, No. 2, p. 200.

As the existence of $A(x, y, s, t)$ satisfying (24) is not assured, we introduce a transformation $A[f(x, s)]$, where f is of the form $\int\int Tg\overline{T}$, such that if

$$f_1(x, s) = A[f(x, s)]$$

then

$$(27) \quad \int\int f_1(x, s)\varphi_i(x)\overline{\varphi}_k(s)dxds = \sum_{jl} \frac{a_{ikjl}}{c_{jl}^2} \int\int f(y, t)\varphi_j(y)\overline{\varphi}_l(t)dydt.$$

The relation between the transformed function $A[f(x, s)]$ and $A(x, y, s, t)$ of (24), when the function exists, is given by

$$(28) \qquad A[f(x, s)] = \int\int A(x, y, s, t)f(y, t)dydt,$$

which follows directly from (27).

The transformed function of $g^*(s)\int f^*(y)L(y, x)dy$ exists and is a continuous function, for it may be expressed in the following form, in which the series on the right are uniformly convergent.

$$\begin{aligned}
&A[g^*(s)\int f^*(y)L(y, x)dy] \\
(29) \quad &= \sum_{ik} \frac{\int \psi_i(y)K(y, x)dy \int f(\xi)\psi_i(\xi)d\xi}{\alpha_i^2} \cdot \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2}\overline{\psi}_k(s) \int g(\eta)\overline{\psi}_k(\eta)d\eta \\
&\quad + \sum_{ik} \frac{\psi_i(x)\int f(\xi)\psi_i(\xi)d\xi}{\alpha_i^2} \cdot \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2} \int g(\eta)\overline{\psi}_k(\eta)d\eta \int \overline{\psi}_k(t)M(t, s)dt.
\end{aligned}$$

The second member of this equation may be written as the product of factors less than unity or of the form $\int F\psi_i$ and $\int F_1\overline{\psi}_k$, F and F_1 denoting continuous functions. As a consequence of (3)

$$\sum_i \left[\int F\psi_i\right]^2 = \int\int FTF$$

and

$$\sum_k \left[\int F\overline{\psi}_k\right]^2 = \int\int F_1\overline{T}F_1.$$

Therefore the two series on the right of (29) converge uniformly. To verify (29) multiply it by $\varphi_j(x)\overline{\varphi}_l(s)$, integrate with respect to x and s, and then factor the second member, thus reducing (29) to (27) with $f = g^*\int f^*L$. The transformed function of $f^*\int g^*N$ exists also and is continuous. Therefore $A[w_\alpha^*(x, s)]$, which we shall denote by A_α, exists and is a continuous function. As

$$A[w_\alpha^*(x, s)] = \frac{w_\alpha^*(x, s)}{\lambda_\alpha}$$

because of (25), we prove that $\{w_\alpha^*(x, s)/\lambda_\alpha\}$ is of finite norm by proving $\sum_{\alpha=1}^{\infty} A_\alpha^2$ convergent.

By the series of operations that follow, we derive the inequality

$$\sum_{\alpha=1}^{n} A_\alpha^2 \leq \Phi^2 + \Psi^2,$$

a continuous function, thus showing the convergency of $\sum_{\alpha=1}^{\infty} A_\alpha^2$. From (29) we obtain

$$A\left[g^* \int f^* L\right] = \sum_{ik} \int f\psi_i \int g\bar{\psi}_k \left[\frac{\int \psi_i K}{\alpha_i^2} \cdot \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} \bar{\psi}_k + \frac{\psi_i \int \bar{\psi}_k M}{\alpha_i^2} \cdot \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2}\right]. \tag{31}$$

Since the second member of (31) is in the form $a(b + c)$ and since $2a(b + c) < a^2 + 2(b^2 + c^2)$, it follows that

$$2\left|A\left[g^*(s) \int f^*(y) L(y, x) dy\right]\right| \leq \sum_{ik} \frac{[\int f(y)\psi_i(y) dy]^2 [\int g(\eta)\bar{\psi}_k(\eta) d\eta]^2}{\alpha_i^2} + \Phi^2(x, s), \tag{32}$$

where

$$\Phi^2(x, s) = 2 \sum_{ik} \left[\left(\int \psi_i(y) K(y, x) dy\right)^2 \cdot \frac{\bar{\psi}_k^2(s)}{\alpha_i^2 + \beta_k^2} + \frac{\psi_i^2(x)}{\alpha_i^2}\left(\int \bar{\psi}_k(t) M(t, s) dt\right)^2\right].$$

The uniform convergence of $\sum_k \frac{\bar{\psi}_k^2}{\beta_k^2}$ and $\sum_i \frac{\psi_i^2}{\alpha_i^2}$ is given by Mercer's theorem,†

$$\sum_i \left[\int \psi_i K\right]^2 = \int KK^* \qquad \text{and} \qquad \sum_k \left[\int \bar{\psi}_k M\right]^2 = \int MM^*$$

by (3), hence Φ^2 is a continuous function. Because of (9) and (3), the inequality (32) becomes

$$2|A[g^*(s) \int f^*(y) L(y, x) dy]| \leq \int \int f(x) L^*(x, y) f(y) dx dy \cdot \int g(s) g^*(s) ds + \Phi^2(x, s). \tag{33}$$

† Mercer, "Symmetrizable Functions and their Expansion in Terms of Biorthogonal Functions," *Proceedings of the Royal Society of London*, Series A, Vol. XCVII, p. 409.

It may likewise be shown that

$$(34)\qquad \begin{aligned} &2\,|A[f^*(x)\textstyle\int g^*(t)N(t,s)dt]| \\ &\qquad\qquad \le \textstyle\int\int g(s)N^*(s,t)g(t)dsdt\cdot\int f(x)f^*(x)dx + \Psi^2(x,s), \end{aligned}$$

where Ψ^2 is a continuous function of x and s. From (33) and (34) there results

$$(35)\qquad \begin{aligned} &\Phi^2(x,s) - 2A[\textstyle\int \overline{T}(\eta,s)\int L(y,x)\int T(y,\xi)f(\xi,\eta)d\xi dy d\eta] \\ &\quad + \textstyle\int\int f(x,s)L^*(x,y)f(y,\eta)dxdy\cdot\int\int \overline{T}(\eta,s)d\eta ds \\ &\quad - 2A[\textstyle\int T(\xi,x)\int N(t,s)\int \overline{T}(\eta,t)f(\xi,\eta)d\xi d\eta dt] \\ &\quad + \textstyle\int\int f(x,s)N^*(s,t)f(\xi,t)dsdt\int\int T(\xi,x)d\xi dx + \Psi^2(x,s) \ge 0. \end{aligned}$$

For $f(x,s)$ in (35) substitute $\sum_{\alpha=1}^{n} A_\alpha u_\alpha(x)v_\alpha(s)$, where $A_\alpha = \dfrac{w_\alpha^*(y,t)}{\lambda_\alpha}$. Then

$$\begin{aligned} &2A\left[\int \overline{T}\int L\int Tf\right] + 2A\left[\int T\int N\int \overline{T}f\right] \\ &\qquad\qquad = 2A\left[\sum_\alpha v_\alpha^*\int Lu_\alpha^* + u_\alpha^*\int Nv_\alpha^*\right]\cdot A_\alpha = 2A_\alpha^2, \end{aligned}$$

and

$$\begin{aligned} &\int\int fL^*f\int\int \overline{T} + \int\int fN^*f\int\int T \\ &\qquad = \sum_\alpha A_\alpha^2\left(\int v_\alpha v_\alpha^*\int\int u_\alpha^* L u_\alpha + \int u_\alpha u_\alpha^*\int\int v_\alpha^* N v_\alpha\right) = \sum_\alpha A_\alpha^2. \end{aligned}$$

Therefore (35) reduces to

$$\Phi^2(x,s) - 2\sum_\alpha A_\alpha^2 + \sum_\alpha A_\alpha^2 + \psi^2(x,s) \ge 0,$$

that is,

$$\sum_{\alpha=1}^{n} A_\alpha^2 \le \Phi^2 + \Psi^2.$$

On account of the continuity of $\Phi^2 + \Psi^2$,

$$\sum_{\alpha=1}^{n} A_\alpha^2 \le P,$$

where P denotes a constant, and $\{w_\alpha^*(x,s)/\lambda_\alpha\}$ is of finite norm for all values of x and s in the given intervals.

As f_α is known to be of finite norm (§ 3), we conclude that the series formed from the absolute values of the terms of

$$\sum_\alpha \frac{w_\alpha^*(x,\,s)f_\alpha}{\lambda_\alpha},$$

where $f(x, s)$ is any continuous function, is uniformly convergent. By (23)

$$h(x,\,s) = \int\int [K^*(x,\,y)f(y,\,t)N^*(t,\,s) + L^*(x,\,y)f(y,\,t)M^*(t,\,s)]dydt$$
$$= \sum_\alpha \frac{w_\alpha^*(x,\,s)f_\alpha}{\lambda_\alpha},$$

and the equality

$$\int\int h(y,\,t)u_\alpha(y)v_\alpha(t)dydt = \frac{f_\alpha}{\lambda_\alpha}$$

follows directly from (22), hence

$$(36)\quad h(x,\,s) = \sum_\alpha \frac{w_\alpha^*(x,\,s)f_\alpha}{\lambda_\alpha} = \sum_\alpha w_\alpha^*(x,\,s)\int\int h(y,\,t)u_\alpha(y)v_\alpha(t)dydt.$$

These results may be expressed as a theorem:

THEOREM 2: *If K, L, M, and N are continuous real functions such that K^*, L^*, M^*, and N^* are symmetric, if L and N are positive definite, and if $KN + LM \not\equiv 0$, any function $h(x, s)$ that can be expressed in the form*

$$h(x,\,s) = \int\int[K^*(x,\,y)f(y,\,t)N^*(t,\,s) + L^*(x,\,y)f(y,\,t)M^*(t,\,s)]dydt,$$

where $f(x, s)$ is any continuous function of x and s, may be expanded into the absolutely uniformly convergent series

$$h(x,\,s) = \sum_\alpha w_\alpha^*(x,\,s)\int\int h(y,\,t)u_\alpha(y)v_\alpha(t)dydt.$$

Equation (36) may be written in the form

$$\int\int\int\int T(x,\xi)[K(\xi,y)f(y,t)N(\eta,t) + L(\xi,y)f(y,t)M(\eta,t)]\overline{T}(\eta,t)d\xi d\eta dydt$$
$$= \sum_\alpha \int\int T(x,\,\xi)w_\alpha(\xi,\,\eta)\overline{T}(\eta,\,t)d\xi d\eta \int\int\int\int [K^*(x,\,y)f(y,\,t)N^*(s,\,t)$$
$$+ L^*(x,\,y)f(y,\,t)M^*(s,\,t)]u_\alpha^*(x)v_\alpha^*(s)dxdsdydt.$$

Inasmuch as T and $\overline{T}$ are definite, it follows that

$$\int\int [K(\xi, y)f(y, t)N(\eta, t) + L(\xi, y)f(y, t)M(\eta, t)]dydt$$

$$= \sum_\alpha w_\alpha(\xi, \eta) \int\int\int\int [K(\xi_1, y)f(y, t)N(\eta_1, t) + L(\xi_1, y)f(y, t)M(\eta_1, t)]dydt \cdot u_\alpha^*(\xi_1)v_\alpha^*(\eta_1)d\xi_1 d\eta_1$$

if the series is uniformly convergent. That is, if

$$g(x, s) = \int\int[K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)]dydt, \tag{37}$$

then

$$g(x, s) = \sum_\alpha w_\alpha(x, s) \int\int g(\xi, \eta)u_\alpha^*(\xi)v_\alpha^*(\eta)d\xi d\eta \tag{38}$$

if the series converges uniformly.

5. **Special Kernels.**—We shall now consider as special cases the equations (1) and (2) when K, L, M, and N are symmetrizable kernels of two important types.

Case I: $K(x, y) = \int \boldsymbol{K}(x, \xi)T(\xi, y)d\xi$, $\quad L(x, y) = \int \boldsymbol{L}(x, \xi)T(\xi, y)d\xi$,

$M(s, t) = \int \boldsymbol{M}(s, \eta)\overline{T}(\eta, t)d\eta$, $\quad N(s, t) = \int \boldsymbol{N}(s, \eta)\overline{T}(\eta, t)d\eta$,

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are symmetric, and $\boldsymbol{L}$, $\boldsymbol{N}$, T, and $\overline{T}$ are symmetric and positive definite.

Case II: $K(x, y) = a(x)T(x, y)$, $\quad L(x, y) = b(x)T(x, y)$,

$M(s, t) = \bar{a}(s)\overline{T}(s, t)$, $\quad N(s, t) = \bar{b}(s)\overline{T}(s, t)$,

where $a(x)$, $b(x)$, $\bar{a}(s)$, $\bar{b}(s)$ are continuous functions and $b(x) > 0$, $\bar{b}(s) > 0$ for x and s in their respective intervals. The functions $T(x, y)$ and $\overline{T}(s, t)$ are symmetric and positive definite. The kernels L and N have only positive characteristic numbers.

Expansion Theorem for Case I.—For this case it will be shown that the series in (38) is uniformly convergent. Substituting for g in the second member of (38) the expression given by (37), we obtain

$$g(x, s) = \sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha} \int\int f(y, t)w_\alpha^*(y, t)dydt. \tag{39}$$

The immediate problem is to show that for Case I $\{w_\alpha/\lambda_\alpha\}$ is of finite norm. We introduce a transformation B such that

$$\int\int T(x, \xi)B[f(x, s)]\overline{T}(\eta, s)dxds = A[T(x, \xi)f(x, s)\overline{T}(\eta, s)], \tag{40}$$

A denoting the transformation defined by (27). The transformed function $B[g(s)\int L(x, y)f(y)dy]$ exists and is a continuous function, for it may be

expressed in the following form by series that are uniformly convergent,

$$B\left[g\int Lf\right]=\sum_{ik}\int K\varphi_i\int f\psi_i\frac{\varphi_k}{\alpha_i^2+\beta_k^2}\int g\bar{\psi}_k \qquad (41)$$
$$+\sum_{ik}\frac{\varphi_i}{\alpha_i^2}\int f\psi_i\frac{\beta_k^2}{\alpha_i^2+\beta_k^2}\int g\bar{\psi}_k\int M\varphi_k.$$

In the second member of (41), $\int K\varphi_i = \int\int KT\varphi_i = \int K\psi_i$, and $\{\int K\psi_i\}$ is of finite norm. Likewise $\{\int f\psi_i\}$, $\{\int g\bar{\psi}_k\}$, and $\{\int M\bar{\varphi}_k\} = \{\int M\bar{\psi}_k\}$ are of finite norm. If we consider the kernel N as a function symmetrizable on the right by the function N, which is of positive type, by Mercer's results* we have the expansion

$$\int N(s,\eta)N(\eta,t)d\eta=\sum_n\frac{\varphi_n(s)\bar{\varphi}_n(t)}{\beta_n^4},$$

and the uniform convergence of the series on the right follows from the uniform convergence of the series in Mercer's expansion for $N(s, t)$.† We conclude, therefore, that $\{\bar{\varphi}_k/\beta_k^2\}$ is of finite norm. Similarly, in the expansion

$$\int L(x,\xi)L(\xi,y)d\xi=\sum_n\frac{\varphi_n(x)\varphi_n(y)}{\alpha_n^4}$$

the series is uniformly convergent, and $\{\varphi_i(x)/\alpha_i^2\}$ is of finite norm. Hence, the two series in (41) are uniformly convergent and represent a continuous function. The transformed function $B[f(x)\int N(s, t)g(t)dt]$ exists also, and is continuous. We have then

$$\int\int T(x,p)B[g(s)\int L(x,y)f(y)dy+f(x)\int N(s,t)g(t)dt]\bar{T}(s,q)dxds$$
$$=A[g^*(q)\int L(y,p)f^*(y)dy+f^*(p)\int N(t,q)g^*(t)dt].$$

When $f = u_\alpha$ and $g = v_\alpha$, this becomes

$$\int\int T(x,p)B[v_\alpha(s)\int L(x,y)u_\alpha(y)dy$$
$$+u_\alpha(x)\int N(s,t)v_\alpha(t)dt]\bar{T}(s,q)dxds=A[w_\alpha^*(p,q)].$$

* Mercer, loc. cit., p. 409. For a kernel k symmetrizable on the right by γ', Mercer gives the expansion

$$\int_a^b k(s,x)\gamma'(x,t)dx=\sum_n\frac{\varphi_n(s)\varphi_n(t)}{\lambda_n\mu_n'},$$

where $\{\varphi_n, \psi_n\}$ is the complete biorthogonal system defined by $\varphi_n = \lambda_n\int k\varphi_n$ and $\psi_n = \lambda_n\int\psi_n k$, and

$$\mu_n'=\frac{1}{\int\int\gamma'(s,t)\psi_n(s)\psi_n(t)dsdt}.$$

† Mercer, loc. cit., p. 409. In Mercer's notation,

$$N=\gamma'(s,t)=\sum_n\frac{\varphi_n(s)\varphi_n(t)}{\mu_n'}+\sum_n\frac{\xi_n'(s)\xi_n'(t)}{\nu_n'}.$$

Substituting for $A[w^*_\alpha(p, q)]$ from (30), and reducing, we obtain

$$B[w_\alpha(x, s)] = \frac{w_\alpha(x, s)}{\lambda_\alpha}. \tag{42}$$

By a process similar to that used in passing from (29) to (32), the following inequality is obtained from (41),

$$2\left|B\left[g(s)\int L(x, y)f(y)dy\right]\right| \leq \sum_{i,k}\frac{[\int f(\xi)\psi_i(\xi)d\xi]^2[\int g(\eta)\overline{\psi}_k(\eta)d\eta]^2}{\alpha_i^2} + \Phi_1^2(x, s), \tag{43}$$

where

$$\Phi_1^2(x, s) = 2\sum_{i,k}\left[\left(\int K(x, y)\varphi_i(y)dy\right)^2\frac{\overline{\varphi}_k^2(s)}{\alpha_i^2+\beta_k^2} + \frac{\varphi_i^2(x)}{\alpha_i^2}\left(\int M(s, t)\overline{\varphi}_k(t)dt\right)^2\right],$$

a continuous function, as the series on the right is uniformly convergent. The uniform convergence of $\sum_i(\int K\varphi_i)^2$ and $\sum_k(\int M\overline{\varphi}_k)^2$ for the special kernels under consideration has already been noted; the uniform convergence of $\sum_i[\varphi_i^2(x)/\alpha_i^2]$ and $\sum_k[\overline{\varphi}_k^2(s)/\beta_k^2]$ results from Mercer's theorem,* if we regard N and L as symmetrizable on the right by N and L, respectively. The inequality (43) may be put into the form

$$2|B[g\int Lf] \leq \int\int fL^*f\int gg^* + \Phi_1^2, \tag{44}$$

and for $B[f\int Ng]$ there is a corresponding inequality,

$$2|B[f\int Ng] \leq \int\int gN^*g\int ff^* + \Psi_1^2, \tag{45}$$

* Mercer, loc. cit., p. 409. The expansion referred to is

$$\gamma'(s, t) = \frac{\sum_n \varphi_n(s)\varphi_n(t)}{\mu_n'} + \sum_n\frac{\xi_n'(s)\xi_n'(t)}{\nu_n'},$$

in which each series is uniformly convergent when $\gamma'(s, s)$ is a continuous function of s in the interval considered. In the notation here used, the first series in the corresponding expansion for L is

$$\sum_n\frac{\varphi_n(x)\varphi_n(y)}{\alpha_n^2}.$$

Hence $\sum_n \varphi_n^2(x)/\alpha_n^2$ is uniformly convergent. Similarly, from the expansion for N, we have the uniform convergence of $\sum_n\overline{\varphi}_n(s)\overline{\varphi}_n(t)/\beta_k^2$.

in which Ψ_1^2 represents a continuous function of x and s. From (44) and (45) there results the relation

$$\begin{aligned}\Phi_1^2(x, s) - 2B[\textstyle\int L(x, y)f(y, s)dy] \\ + \int\int f(x, s)L^*(x, y)f(y, \eta)dxdy\int\int \overline{T}(\eta, s)d\eta ds \\ + \Psi_1^2(x, s) - 2B[\textstyle\int N(s, t)f(t, x)dt] \\ + \int\int f(x, s)N^*(s, t)f(\xi, t)dsdt\int\int T(\xi, x)d\xi dx \geq 0.\end{aligned} \tag{46}$$

If

$$f(y, t) = \sum_{\alpha=1}^{n} B_\alpha u_\alpha(y)v_\alpha(t) \tag{47}$$

and

$$B_\alpha = \frac{w_\alpha(x, s)}{\lambda_\alpha},$$

then by (42)

$$B_\alpha = B[w_\alpha(x, s)].$$

With the substitution for f of the function given by (47), the inequality (46) reduces to

$$\Phi_1^2(x, s) - 2\sum_\alpha B_\alpha^2 + \sum_\alpha B_\alpha^2 + \Psi_1^2(x, s) \geq 0$$

or

$$\sum_{\alpha=1}^{n} B_\alpha^2 \leq \Phi_1^2 + \Psi_1^2.$$

Therefore $\{w_\alpha(x, s)/\lambda_\alpha\}$ is of finite norm. From this result and the fact that $\{f_i\}$ is of finite norm, we conclude that the series

$$\sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha}\int\int f(y, t)w_\alpha^*(y, t)dydt$$

is uniformly convergent. The following theorem has then been proved:

THEOREM 3: *If the kernels of equations* (1) *have the form* $K = \int \mathbf{K}T$, $L = \int \mathbf{L}T$, $M = \int \mathbf{M}\overline{T}$, $N = \int \mathbf{N}\overline{T}$, *where* $\mathbf{L}$, $\mathbf{N}$, T, *and* $\overline{T}$ *are symmetric and positive definite, and* $\mathbf{K}$ *and* $\mathbf{M}$ *are symmetric, a function* $g(x, s)$ *which has the form*

$$g(x, s) = \int\int[K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)]dydt,$$

$f(y, t)$ *being any continuous function, may be expanded into the uniformly convergent series*

$$g(x, s) = \sum_\alpha w_\alpha(x, s)\int\int g(\xi, \eta)u_\alpha^*(\xi)v_\alpha^*(\eta)d\xi d\eta,$$

where w_α is the function defined by (18) *and u^*_α, v^*_α are the solutions of the adjoint system* (2).

Expansion Theorems for Case II.—Before taking up the question of the expansion of arbitrary functions, we note certain reductions that are a consequence of the special form of the kernels. The equations now considered are

$$(48)\qquad \begin{aligned} u(x) &= \lambda \int a(x)\,T(x, y)u(y)dy + \mu \int b(x)\,T(x, y)u(y)dy, \\ v(s) &= \lambda \int \bar{a}(s)\,\overline{T}(s, t)v(t)dt - \mu \int \bar{b}(s)\,\overline{T}(s, t)v(t)dt, \end{aligned}$$

which may be written

$$(49)\qquad \begin{aligned} u(x) &= [\lambda a(x) + \mu b(x)]u^*(x), \\ v(s) &= [\lambda \bar{a}(s) - \mu \bar{b}(s)]v^*(s). \end{aligned}$$

Substituting from (49) in $w_\alpha(x, s)$, we obtain

$$(50)\qquad \begin{aligned} w_\alpha(x, s) &= \lambda_\alpha[a(x)\bar{b}(s) + b(x)\bar{a}(s)]u^*_\alpha(x)v^*_\alpha(s) \\ &= \lambda_\alpha F(x, s)u^*_\alpha(x)v^*_\alpha(s), \end{aligned}$$

where

$$F(x, s) = a(x)\bar{b}(s) + b(x)\bar{a}(s).$$

The equations (8) which define the biorthogonal systems $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$ give the relations

$$\varphi_i(x) = \alpha_i^2 b(x)\psi_i(x), \qquad \overline{\varphi}_k(s) = \beta_k^2 \bar{b}(s)\overline{\psi}_k(s).$$

It follows that

$$\alpha_i^2 \int b\psi_i\psi_j = \delta_{ij}$$

and therefore

$$\{\chi_i\} = \{\alpha_i \sqrt{b}\,\psi_i\}$$

is a closed normalized orthogonal system of functions. Similarly

$$\{\overline{\chi}_k\} = \{\beta_k \sqrt{\bar{b}}\,\overline{\psi}_k\}$$

is a closed normalized orthogonal system. We pass from the biorthogonal systems to the orthogonal by means of the relations

$$\varphi_i = \alpha_i \sqrt{b}\,\chi_i, \qquad \psi_i = \frac{1}{\alpha_i}\frac{\chi_i}{\sqrt{b}},$$

$$\overline{\varphi}_k = \beta_k \sqrt{\bar{b}}\,\overline{\chi}_k, \qquad \overline{\psi}_k = \frac{1}{\beta_k}\frac{\overline{\chi}_k}{\sqrt{\bar{b}}}.$$

From (8)

$$\chi_i(x) = \alpha_i^2 \int \sqrt{b(x)}\, T(x, \xi) \sqrt{b(\xi)}\, \chi_i(\xi) d\xi,$$

$$\overline{\chi}_k(s) = \beta_k^2 \int \sqrt{\overline{b}(s)}\, \overline{T}(s, \eta) \sqrt{\overline{b}(\eta)}\, \overline{\chi}_k(\eta) d\eta.$$

It follows from (14) and (3) that

$$\int \int u_\alpha^*(x) v_\alpha^*(s) \cdot \lambda_\beta F(x, s) u_\beta^*(x) v_\beta^*(s) dx ds = \delta_{\alpha\beta},$$

hence

$$\lambda_\alpha \int \int F(x, s) u_\alpha^{*2}(x) v_\alpha^{*2}(s) dx ds = 1,$$

which shows that $\lambda_\alpha > 0$ if $F(x, s) > 0$. Therefore, when $F(x, s) > 0$ and $\lambda_\alpha > 0$, the functions $\{ \sqrt{\lambda_\alpha} \sqrt{F(x, s)} u_\alpha^*(x) v_\alpha^*(s) \} = \{ \zeta_\alpha \}$ form an orthogonal system.

We now proceed to prove that the series (39) is uniformly convergent when the kernels are of the type under consideration and $F(x, s) > 0$. As

$$\sum_\alpha{}' \frac{w_\alpha(x, s)}{\lambda_\alpha} \int \int f(y, t) w_\alpha^*(y, t) dy dt$$
$$= \sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha^{3/2}} \cdot \sqrt{\lambda_\alpha} \int \int f(y, t) w_\alpha^*(y, t) dy dt,$$

the series is uniformly convergent if $\{ w_\alpha(\xi, \eta)/\lambda_\alpha^{3/2} \}$ and $\{ \sqrt{\lambda_\alpha} \int \int f w_\alpha^* \}$ are of finite norm. Let B denote the transformation defined by (40). In

$$B[f(x, s)] = \sum_\alpha \int K\varphi_i \int \int \psi_i f \overline{\psi}_k \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} \overline{\varphi}_k$$
$$+ \sum_\alpha \frac{\varphi_i}{\alpha_i^2} \int \int \psi_i f \overline{\psi}_k \int M \overline{\varphi}_k \cdot \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2},$$

substitute aT and $\bar{a}\overline{T}$ for K and M, respectively, and change from the biorthogonal functions $\{\varphi_i, \psi_i\}$, $\{\overline{\varphi}_k, \overline{\psi}_k\}$ to the orthogonal functions $\{\chi_i\}$, $\{\overline{\chi}_k\}$, thus obtaining

$$B[f(x, s)] = \sum_\alpha \frac{\chi_i(\xi) f(\xi, \eta) \overline{\chi}_k(\eta) d\xi d\eta}{\sqrt{b(\xi)} \sqrt{\overline{b}(\eta)}} \cdot \frac{F(x, s)}{\sqrt{b(x)} \sqrt{\overline{b}(s)}} \cdot \frac{\chi_i(x) \overline{\chi}_k(s)}{\alpha_i^2 + \beta_k^2}.$$

Then

$$2 | B[f(x, s)] | \leq \int \int \frac{f^2(\xi, \eta) d\xi d\eta}{b(\xi) \overline{b}(\eta)} + \Pi^2(x, s),$$

where

$$\Pi^2(x, s) = \frac{F^2(x, s)}{b(x) \overline{b}(s)} \cdot \sum_\alpha \frac{\chi_i^2(x)}{\alpha_i^2} \cdot \frac{\overline{\chi}_k^2(s)}{\beta_k^2}.$$

Since

$$\int\int f(x)\sqrt{b(x)}\,T(x, y)\sqrt{b(y)}f(y)dxdy \geq 0$$

the series $\sum_i [\chi_i^2(x)/\alpha_i^2]$ is uniformly convergent.* The series $\sum_k [\chi_k^2(s)/\beta_k^2]$ is also uniformly convergent, hence we conclude that $\Pi^2(x, s)$ is a continuous function. Let us assume that $F(x, s)$ satisfies the condition $0 < F(x, s) < b(x)\bar{b}(s)$, a condition that may be imposed without limiting the problem. Then

$$\int\int\frac{f^2(\xi, \eta)}{b(\xi)\bar{b}(\eta)}d\xi d\eta \leq \int\int\frac{f^2(\xi, \eta)}{F(\xi, \eta)}d\xi d\eta,$$

and it follows that

$$2|B[f(x, s)]| \leq \int\int\frac{f^2(\xi, \eta)}{F(\xi, \eta)}d\xi d\eta + \Pi^2(x, s). \tag{51}$$

If

$$f(x, s) = \sum_{\alpha=1}^{n} C_\alpha\sqrt{\lambda_\alpha}F(x, s)u_\alpha^*(x)v_\alpha^*(s),$$

where

$$C_\alpha = \frac{w_\alpha(y, t)}{\lambda_\alpha^{3/2}} = B\left[\frac{w_\alpha(y, t)}{\sqrt{\lambda_\alpha}}\right]$$

the inequality (51) becomes

$$\Pi^2(x, s) - 2\sum_{\alpha=1}^{n} C_\alpha B[\sqrt{\lambda_\alpha}F(x, s)u_\alpha^*(x)v_\alpha^*(s)]$$
$$+\sum_{\alpha,\beta=1}^{n} C_\alpha C_\beta\int\int\sqrt{\lambda_\alpha}\sqrt{\lambda_\beta}F(\xi, \eta)u_\alpha^*(\xi)v_\alpha^*(\eta)u_\beta^*(\xi)v_\beta^*(\eta)d\xi d\eta \geq 0.$$

This reduces to

$$\Pi^2(x, s) - 2\sum_{\alpha=1}^{n} C_\alpha\cdot B\left[\frac{w_\alpha(x, s)}{\sqrt{\lambda_\alpha}}\right] + \sum_{\alpha=1}^{n} C_\alpha^2 \geq 0$$

or

$$\sum_{\alpha=1}^{n} C_\alpha^2 \leq \Pi^2(x, s).$$

Hence $\{w_\alpha(y, t)/\lambda_\alpha^{3/2}\}$ is of finite norm. To show $\{\sqrt{\lambda_\alpha}\int\int f(x, s)w_\alpha^*(x, s)dxds\}$ of finite norm, substitute for $w_\alpha^*(x, s)$ from (15) and exhibit ζ_α as a factor,—

* Mercer, loc. cit., p. 407.

of each term, thus obtaining

$$\sqrt{\lambda_\alpha}\int\int f(x, s)w_\alpha^*(x, s)dxds$$

$$= \int\int\int\frac{f(x, s)N(s, t)}{\sqrt{F(x, t)}}\,ds\,\sqrt{\lambda_\alpha}\sqrt{F(x, t)}u_\alpha^*(x)v_\alpha^*(t)dxdt$$

$$+ \int\int\int\frac{f(x, s)L(x, y)}{\sqrt{F(y, s)}}\,dx\,\sqrt{\lambda_\alpha}\sqrt{F(s, y)}u_\alpha^*(y)v_\alpha^*(s)dyds$$

$$= \int\int h(x, t)\zeta_\alpha(x, t)dxdt,$$

where

$$h(x, t) = \int\frac{f(x, s)N(s, t)}{\sqrt{F(x, t)}}\,ds + \int\frac{f(x, s)L(x, y)}{\sqrt{F(y, s)}}\,dx.$$

But

$$\sum_\alpha\left[\int\int h(x, t)\zeta_\alpha(x, t)dxdt\right]^2 \leqq \int\int[h(x, t)]^2dxdt;$$

therefore

$$\sum_\alpha\left[\sqrt{\lambda_\alpha}\int\int f(x, s)w_\alpha^*(x, s)dxds\right]^2 \leq 2\int\int[h(x, t)]^2dxdt;$$

that is, $\{\sqrt{\lambda_\alpha}\int\int f(x, s)w_\alpha^*(x, s)dxds\}$ is of finite norm. Since each of the sequences is of finite norm, the series (47) is uniformly convergent.

The function to which the series converges, $\int\int[K(x, y)f(y, t)N(s, t) + L(x,y)f(y,t)M(s,t)]dydt$, takes the form $F(x,s)\int\int T(x,y)f(y,t)\overline{T}(t,s)dydt$ for the special kernels considered. Hence, with the assumption $F > 0$, we have obtained the expansion

$$F(x, s)\int\int T(x, y)f(y, t)\overline{T}(t, s)dydt = \sum_\alpha\frac{w_\alpha(x, s)}{\lambda_\alpha}\int\int f(\xi, \eta)w_\alpha^*(\xi, \eta)d\xi d\eta.$$

Because of (50), this reduces to

$$\int\int T(x, y)f(y, t)\overline{T}(t, s)dydt = \sum_\alpha u_\alpha^*(x)v_\alpha^*(s)\int\int f(\xi, \eta)w_\alpha^*(\xi, \eta)d\xi d\eta.$$

These results give the following theorem:

THEOREM 4: *If the kernels of system* (1) *have the form* $K(x, y) = a(x)T(x, y)$, $L(x, y) = b(x)T(x, y)$, $M(s, t) = \bar{a}(s)\overline{T}(s, t)$, $N(s, t) = \bar{b}(s)\overline{T}(s, t)$, *where* a, b, $\bar{a}$, $\bar{b}$ *are continuous functions,* $b > 0$, $\bar{b} > 0$, $a\bar{b} + \bar{a}b > 0$ *and* T, $\overline{T}$ *are symmetric positive definite kernels, a function* $g(x, s)$ *which is expressible as* $\int\int T(x, y)f(y, t)\overline{T}(t, s)dydt$, $f(y, t)$ *denoting a continuous function of* y

and t, may be expanded into the uniformly convergent series

$$g(x, s) = \sum_\alpha u^*_\alpha(x) v^*_\alpha(s) \int\int f(\xi, \eta) w^*_\alpha(\xi, \eta) d\xi d\eta$$
$$= \sum_\alpha u^*_\alpha(x) v^*_\alpha(s) \int\int g(y, t) w_\alpha(y, t) dy dt,$$

where u^*_α, v^*_α *are the solutions of the adjoint system* (2) *and* w^*_α *and* w_α *are the functions defined by* (15) *and* (18), *respectively.*

We now show without any assumption as to the positive character of $F(x, s)$ that the series

$$\sum_\alpha \frac{w_\alpha(x, s)}{\lambda^2_\alpha} \int\int f(y, t) w^*_\alpha(y, t) dy dt$$

is uniformly convergent. Using again the transformation B, we find that the transformed function $B[B[g(s) \int L(x, y) f(y) dy]]$ has the following form:

$$B[B[g \int Lf]] = \sum_{ik} \sum_{jl} \frac{\int f(y)\psi_i(y)dy \int g(\eta)\bar{\psi}_l(\eta) d\eta}{\alpha_j}$$
$$\times \int\int \psi_i(\xi)\bar{\psi}_k(\eta) \frac{F(\xi, \eta)}{b(\xi)\bar{b}(\eta)} \cdot \frac{\varphi_j(\xi)\bar{\varphi}_l(\eta)}{\alpha_j(\alpha^2_j + \beta^2_l)} d\xi d\eta \frac{F(x, s)}{b(x)\bar{b}(s)} \cdot \frac{\varphi_i(x)\bar{\varphi}_k(s)}{\alpha^2_i + \beta^2_k}.$$

Upon the introduction of the orthogonal functions $\{\chi_i\}$ and $\{\bar{\chi}_k\}$ this becomes

$$B[B[g \int Lf]] = \sum_{jl} \frac{\int f(y)\psi_i(y)dy \int g(\eta)\bar{\psi}_l(\eta) d\eta}{\alpha_j}$$

$$(52) \qquad \times \left[\sum_i \frac{\int \chi_i(\xi) \dfrac{a(\xi)}{b(\xi)} \chi_j(\xi) d\xi}{\alpha^2_j + \beta^2_l} \sqrt{b(x)\bar{b}(s)} \frac{\beta_l \chi_i(x) \bar{\chi}_l(s)}{\alpha^2_i + \beta^2_l} \right.$$
$$\left. + \sum_k \frac{\int \bar{\chi}_k(\eta) \dfrac{\bar{a}(\eta)}{\bar{b}(\eta)} \bar{\chi}_l(\eta)}{\alpha^2_j + \beta^2_l} \sqrt{b(x)\bar{b}(s)} \frac{\beta_l \chi_j(x) \bar{\chi}_k(s)}{\alpha^2_j + \beta^2_k} \right].$$

By a series of operations similar to those employed in passing from (31) to (33), there result from equation (52) and the corresponding equation for $B[B[g \int Lf]]$ the inequalities

$$(53) \qquad 2|B[B[g \int Lf]]| \leq \int\int f L^* f \int\int g g^* + \Pi^2_1,$$

$$(54) \qquad 2|B[B[f \int Ng]]| \leq \int\int g N^* g \int f f^* + \Pi^2_2,$$

Π^2_1 and Π^2_2 denoting continuous functions. As a consequence it follows that

$$\Pi^2_1(x, s) + \int\int f(x, s) L^*(x, y) f(y, s) dx ds \int\int \bar{T}(\eta, s) d\eta ds$$
$$+ \Pi^2_2(x, s) + \int\int f(x, s) N^*(s, t) f(\xi, t) ds dt \int\int T(\xi, x) d\xi dx$$
$$- 2B[B[L(x, y) f(y, s) dy]] - 2B[B[N(s, t) f(t, x) dt]] \geq 0.$$

Let

$$f(y, t) = \sum_{\alpha=1}^{n} B_\alpha u_\alpha(y) v_\alpha(t),$$

where

$$B_\alpha = \frac{w_\alpha(x, s)}{\lambda_\alpha^2}.$$

The substitution of this value of f reduces the preceding inequality to

$$\Pi_1^2(x, s) + \Pi_2^2(x, s) + \sum_{\alpha=1}^{n} B_\alpha^2 - 2\sum_{\alpha=1}^{n} B_\alpha^2 \geq 0,$$

or

$$\sum_{\alpha=1}^{n} B_\alpha^2 \leq \Pi_1^2 + \Pi_2^2.$$

Hence $\{w_\alpha/\lambda_\alpha^2\}$ is of finite norm and the series

$$\sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha^2} \int\int f(y, t) w_\alpha^*(y, t)\, dydt$$

is uniformly convergent.

The next step is to determine the form of the function represented by the expansion. Equation (21) gives the relation

$$a_{ikjl} = \sum_\alpha \frac{c_{ik}^2 x_{\alpha ik}^* \cdot c_{jl}^2 x_{\alpha jl}^*}{\lambda_\alpha},$$

which, in turn, gives

$$\sum_{j_1 l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot a_{j_1l_1jl} = \sum_\alpha \frac{\sum_{j_1l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot c_{j_1l_1}^2 x_{\alpha j_1l_1}^* \cdot c_{jl}^2 x_{\alpha jl}^*}{\lambda_\alpha}.$$

By (26), this reduces to

$$\sum_{j_1 l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot a_{j_1l_1jl} = \sum_\alpha \frac{c_{ik}^2 x_{\alpha ik} \cdot c_{jl}^2 x_{\alpha jl}}{\lambda_\alpha^2}$$

or

$$\sum_{j_1 l_1} \frac{a_{ikj_1l_1}}{c_{j_1l_1}^2} \cdot a_{j_1l_1jl} = \sum_\alpha \frac{w_{\alpha ik} w_{\alpha jl}}{\lambda_\alpha^2},$$

and this by means of the relation

$$w_{\alpha ik} = \int\int \psi_i \bar{\psi}_k w_\alpha = \int\int \varphi_i \bar{\varphi}_k w_\alpha^*$$

and the definition of the transformation B gives the equality

$$B\left[\int\int \{K(x, y) f(y, t) N(s, t) + L(x, y) f(y, t) M(s, t)\}\, dydt\right]$$
$$= \sum_\alpha \frac{w_\alpha(xs)}{\lambda_\alpha^2} \int\int f(y, t) w_\alpha^*(y, t)\, dydt.$$

This completes the proof of

THEOREM 5: *Any continuous function $h(x, s)$ that is expressible in the form*

$$h(x, s) = B[\int\int \{K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)\}dydt],$$

where K, L, M, and N are kernels of the type of Case II and $f(y, t)$ is a continuous function of y and t, may be expanded into the uniformly convergent series

$$\begin{aligned} h(x, s) &= \sum_\alpha \frac{w_\alpha(x, s)}{\lambda_\alpha^2} \int\int f(y, t) w_\alpha^*(y, t)dydt \\ &= \sum_\alpha w_\alpha(x, s) \int\int h(y, t)u_\alpha^*(y)v_\alpha^*(t)dydt \\ &= F(x, s) \sum_\alpha u_\alpha^*(x)v_\alpha^*(s) \cdot \lambda_\alpha \int\int h(y, t)u_\alpha^*(y)v_\alpha^*(t)dydt, \end{aligned}$$

where w_α and w_α^* are the functions defined by (18) *and* (15) *respectively, and u_α^*, v_α^* are the solutions of the adjoint system* (2) *with the kernels of Case II.*

Let $T(x, y)$ and $\overline{T}(s, t)$ denote the Green's functions that vanish at the ends of the intervals considered, of the differential expressions

$d\left(p\dfrac{du^*}{dx}\right) \Big/ dx$ and $d\left(\pi\dfrac{dv^*}{ds}\right) \Big/ ds$, respectively, where $p > 0$ and $\pi > 0$.

The system of differential equations (b) (§ 1) with the given boundary conditions is equivalent to a system of integral equations of the form

$$u^*(x) = \lambda \int_a^b T(x, y)a(y)u^*(y)dy + \mu \int_a^b T(x, y)b(y)u^*(y)dy,$$

$$v^*(s) = \lambda \int_a^b \overline{T}(s, t)\bar{a}(t)v^*(t)dt - \mu \int_c^d \overline{T}(s, t)\bar{b}(t)v^*(t)dt,$$

a, b, $\bar{a}$, $\bar{b}$ denoting analytic functions, and this is exactly the adjoint system of (48). By Theorem 4 any continuous function having continuous first and second derivatives and satisfying the given boundary conditions may be expanded into a uniformly convergent series in terms of the solutions u_α^*, v_α^* of the differential equations, if $F(x, s) > 0$.

If $F(x, s)$ is not everywhere positive, it follows from Theorem 5 that every function $g(x, s)$ continuous and with continuous derivatives of the first four orders, satisfying certain boundary conditions, may be expanded into a uniformly convergent series in terms of $u_\alpha^* v_\alpha^*$. For when $g(x, s)$ is subject to the above conditions, there exists a function $k(x, s)$ continuous and with continuous first and second derivatives, satisfying certain boundary conditions, such that

$$g(x, s) = \sqrt{b(x)}\int T(x, y)b(y)dy \cdot k(x, s)\sqrt{\bar{b}(s)}\int \overline{T}(s, t)\bar{b}(t)dt.$$

Hence

$$\int\int g(x, s)\chi_i(x)\overline{\chi}_k(s)dxds = \frac{\int\int \chi_i(x)k(x, s)\overline{\chi}_k(s)dxds}{\alpha_i^2\beta_k^2}.$$

Further a function $f(y, t)$ can be found such that

$$\int\int T(\xi, y)f(y, t)\overline{T}(\eta, t)dydt$$
$$= b(\xi)\sqrt{\overline{b}(\eta)}\int \overline{T}(\xi, \eta_1)b(\eta_1)k(\eta, \eta_1)d\eta_1$$
$$+ b(\xi)\overline{b}(\eta)\int T(\eta_1, \eta)\sqrt{\overline{b}(\eta_1)}k(\xi, \eta_1)d\eta_1$$

and therefore

$$\int\int\int\int T(\xi, y)f(y, t)\overline{T}(\eta, t)dydt\cdot\frac{F(\xi, \eta)}{\sqrt{b(\xi)}\sqrt{\overline{b}(\eta)}}\cdot\frac{\chi_i(\xi)\overline{\chi}_k(\eta)}{\alpha_i^2+\beta_k^2}$$
$$= \int\int \chi_i(\xi)k(\xi, \eta)\overline{\chi}_k(\eta)d\xi d\eta\cdot\frac{\alpha_i^2+\beta_k^2}{\alpha_i^2\beta_k^2}\cdot\frac{1}{\alpha_i^2+\beta_k^2}$$
$$= \int\int g(\xi, \eta)\chi_i(\xi)\overline{\chi}_k(\eta)d\xi d\eta.$$

It follows that

$$B\left[\int\int\{K(x, y)f(y, t)N(s, t) + L(x, y)f(y, t)M(s, t)\}dydt\right.$$
$$= g(x, s)\frac{F(x, s)}{\sqrt{b(x)\overline{b}(s)}};$$

that is, the function g is expressible in the form specified for the function h in Theorem 5.

Thus Hilbert's results for the expansion of an arbitrary function of two variables in terms of the solutions of the differential equations (b) are obtained as a special case of the results stated in Theorems 4 and 5.

THE ASYMPTOTIC EXPANSION OF THE FUNCTIONS $W_{k,m}(z)$ OF WHITTAKER.

BY F. H. MURRAY.

The asymptotic expansion of the functions $W_{k,m}(z)$ has been given* for any sector $|\arg z| < \pi - \epsilon$ where $\epsilon > 0$.

In many applications it is convenient to have also the expansion in a sector including the negative half of the real axis; in this paper it will be shown that if the parameters k and m satisfy certain inequalities, the expansion given by Whittaker remains valid in such a sector. This result is applied in a study of the "croissance" of the solutions of a class of linear differential equations of the second order, forming an extension of an earlier paper by the writer.

1. If $R(k - \frac{1}{2} - m) \leq 0$, the function $W_{k,m}(z)$ is defined by the formula

$$(1) \qquad W_{k,m}(z) = \frac{e^{-(z/2)}z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty t^{-k-1/2+m}\left(1 + \frac{t}{z}\right)^{k-1/2+m} e^{-t}dt \cdot$$

$$(-\pi < \arg z < \pi)$$

and satisfies the differential equation

$$(2) \qquad \frac{d^2W}{dz^2} + \left\{ -\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right\} W = 0.$$

If z is not real, $W_{k,m}(z)$ and $W_{-k,m}(-z)$ are linearly independent solutions of (2); if z is real, the values $W_{k,m}(z + io)$ or $W_{k,m}(z - io)$ can be taken. When z is not real, the asymptotic expansion for $W_{k,m}(z)$ is known; it will be shown that if k, m are real, and

$$(3) \qquad -1 < k - \tfrac{1}{2} + m < 0,$$

this expansion remains valid when z approaches a point on the negative real axis.

If z is not real, and $\lambda = k - \frac{1}{2} + m$,

$$(4) \qquad \begin{aligned} \left(1 + \frac{t}{z}\right)^\lambda &= 1 + \lambda\frac{t}{z} + \frac{\lambda(\lambda - 1)}{2!}\left(\frac{t}{z}\right)^2 + \cdots \\ &+ \frac{\lambda(\lambda - 1)\cdots(\lambda - n + 1)}{n!}\left(\frac{t}{z}\right)^n \\ &+ \frac{\lambda(\lambda - 1)\cdots(\lambda - n)}{n!}\left(1 + \frac{t}{z}\right)^\lambda \int_0^{t/z} u^n(1 + u)^{-1-\lambda}du, \end{aligned}$$

* *Bull. Amer. Math. Soc.*, Vol. X. Whittaker and Watson, "Modern Analysis," Chap. XVI, 2d or 3d edition.

where for convenience a straight line path of integration will be chosen. As in (1), that branch of the function $[1 + (t/z)]^\lambda$ is chosen which is real and positive when z is real and positive. If the right-hand member of (4) is substituted in (1) and the terms integrated separately, the following expansion is obtained:

$$
\begin{aligned}
W_{k,m}(z) = e^{-(z/2)}z^k \Bigg\{ 1 &+ \frac{m^2 - (k - \frac{1}{2})^2}{1!\, z} \\
(5) \qquad &+ \frac{\{m^2 - (k - \frac{1}{2})^2\}\{m^2 - (k - \frac{3}{2})^2\}}{2!\, z^2} + \cdots \\
&+ \frac{\{m^2 - (k - \frac{1}{2})^2\} \cdots \{m^2 - (k - n + \frac{1}{2})^2\}}{n!\, z^n} + R_n(z) \Bigg\},
\end{aligned}
$$

in which

$$
(6) \quad R_n(z) = \frac{\lambda(\lambda-1)\cdots(\lambda-n)}{n!\,\Gamma(-k+\frac{1}{2}+m)} \int_0^\infty t^{\lambda-2k} e^{-t}\left(1+\frac{t}{z}\right)^\lambda \left[\int_0^{t/z} u^n (1+u)^{-1-\lambda} du\right] dt,
$$

$$
n - k - \tfrac{1}{2} + m > 0.
$$

It remains to discuss the remainder term when z is in the neighborhood of a point on the negative half of the real axis. If t is real, and $z = -z'$,

$$
z' = x + iy, \qquad y \neq 0,
$$

$$
(7) \qquad R\left(\frac{t}{z'}\right) = \frac{tx}{|z'|^2} = \frac{t}{t_1}
$$

if

$$
t_1 = \frac{|z'|^2}{x} = |z'| \sec\alpha, \qquad \alpha = \arctan\frac{y}{x}.
$$

Also

$$
\int_0^{t/z'} u^n (1 + u)^{-\lambda-1} du = (-1)^{n-1} \int_0^{t/z'} u^n (1 - u)^{-\lambda-1} du.
$$

Substituting in (6),

$$
(8) \qquad R_n(-z') = (-1)^{n-1} \frac{\lambda(\lambda-1)\cdots(\lambda-n)}{n!\,\Gamma(-k+\frac{1}{2}+m)} [I_1 + I_2],
$$

$$
(9) \qquad
\begin{aligned}
I_1 &= \int_0^{t_1} t^{\lambda-2k} e^{-t}\left(1 - \frac{t}{z'}\right)^\lambda \left[\int_0^{t/z'} u^n (1 - u)^{-\lambda-1} du\right] dt, \\
I_2 &= \int_{t_1}^{\infty} t^{\lambda-2k} e^{-t}\left(1 - \frac{t}{z'}\right)^\lambda \left[\int_0^{t/z'} u^n (1 - u)^{-\lambda-1} du\right] dt.
\end{aligned}
$$

If a straight line path of integration is chosen, and $u = \bar{x} + i\bar{y}$, then since $-\lambda - 1 < 0$,

$$
(10)\qquad
\begin{gathered}
|du| = \sec \alpha d\bar{x}, \qquad |1 - u| \geq |1 - \bar{x}|, \\
|(1-u)^{-\lambda-1}| \leq |(1-\bar{x})^{-\lambda-1}|, \\
\left|1 - \frac{t}{z'}\right| \geq \left|1 - \frac{t}{t_1}\right|, \qquad \left|\left(1 - \frac{t}{z'}\right)^{\lambda}\right| \leq \left|\left(1 - \frac{t}{t_1}\right)^{\lambda}\right|.
\end{gathered}
$$

Consequently* if $t < t_1$,

$$
(11)\qquad
\begin{aligned}
\left|\int_0^{t/z'} u^n (1-u)^{-\lambda-1} du\right| &\leq \left|\frac{t}{z'}\right|^n \sec\alpha \int_0^{t/t_1} (1-\bar{x})^{-\lambda-1} d\bar{x} \\
&\leq \left|\frac{\sec\alpha}{\lambda}\left(\frac{t}{z'}\right)^n\right| \cdot \left|1 - \left(1 - \frac{t}{t_1}\right)^{-\lambda}\right|,
\end{aligned}
$$

from which

$$
(12)\qquad |I_1| \leq \left|\frac{\sec\alpha}{\lambda z'^n}\right| \int^{t_1} t^{\lambda-2k+n} e^{-t} \left|1 - \left(1 - \frac{t}{t_1}\right)^{\lambda}\right| dt.
$$

Or if

$$
I_1' = \int_0^{t_1} t^{\lambda-2k+n} e^{-t} dt, \qquad I_1' < \Gamma(\lambda - 2k + n + 1),
$$

$$
(13)\qquad I_1'' = \int_0^{t_1} t^{\lambda-2k+n} e^{-t} \left(1 - \frac{t}{t_1}\right)^{\lambda} dt,
$$

$$
(14)\qquad |I_1| \leq \left|\frac{\sec\alpha}{\lambda z'^n}\right| [I_1' + I_1''].
$$

To find an upper bound for I_1'', introduce the integrals

$$
J_1 = \int_0^{t_1/2} t^{\lambda-2k+n} e^{-t} (t_1 - t)^{\lambda} dt,
$$

$$
J_2 = \int_{t_1/2}^{t_1} t^{\lambda-2k+n} e^{-t} (t_1 - t)^{\lambda} dt.
$$

Then

$$
(15)\qquad I_1'' = t_1^{-\lambda}(J_1 + J_2).
$$

In J_1,

$$
t_1 - t \geq \frac{t_1}{2}, \qquad (t_1 - t)^{\lambda} \leq \left(\frac{t_1}{2}\right)^{\lambda}.
$$

Consequently

$$
(16)\qquad J_1 < \left(\frac{t_1}{2}\right)^{\lambda} \Gamma(\lambda - 2k + n + 1).
$$

---

* In the following developments if the real power of a positive quantity is indicated, the absolute value will be understood.

Also

$$J_2 < e^{-(t_1/2)} t_1^{\lambda-2k+n} \cdot \int_{t_1/2}^{t_1} (t_1 - t)^\lambda \, dt < \frac{e^{-(t_1/2)} t_1^{2\lambda-2k+n+1}}{2^{\lambda+1}(\lambda+1)}.$$

Hence

$$(17) \qquad |I_1| < \left|\frac{\sec\alpha}{\lambda z'^n}\right| \left\{ \Gamma(\lambda - 2k + n + 1)(1 + 2^{-\lambda}) + \frac{e^{-(t_1/2)} t_1^{\lambda-2k+n+1}}{2^{\lambda+1}(\lambda+1)} \right\}.$$

Since $-1 < -\lambda - 1$, we obtain from (9), (10),

$$(18) \qquad |I_2| \leq \frac{\sec\alpha}{|z'|^n} \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} \left(1 - \frac{t}{t_1}\right)^\lambda \left[\int_0^{t/t_1} |(1-\bar{x})^{-\lambda-1}| \, d\bar{x} \right] dt.$$

Also,

$$\int_0^{t/t_1} |(1-\bar{x})^{-\lambda-1}| \, d\bar{x} = \int_0^1 (\cdots) d\bar{x} + \int_1^{t/t_1} (\cdots) d\bar{x}$$
$$= \frac{1}{|\lambda|}\left[1 + \left(\frac{t}{t_1} - 1\right)^{-\lambda}\right].$$

Consequently from (18),

$$(19) \qquad \begin{aligned} |I_2| &\leq \frac{\sec\alpha}{|z'|^n|\lambda|} \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} \left[1 + \left(\frac{t}{t_1} - 1\right)^\lambda\right] dt \\ &\leq \frac{\sec\alpha}{|z'|^n|\lambda|} [I_2' + I_2''], \end{aligned}$$

$$(20) \qquad I_2' = \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} dt < \Gamma(\lambda - 2k + n + 1),$$

$$(21) \qquad I_2'' = t_1^{-\lambda} \int_{t_1}^{\infty} t^{\lambda-2k+n} e^{-t} (t - t_1)^\lambda dt.$$

Let $\bar{t} = t - t_1$.

$$(22) \qquad \begin{aligned} I_2'' &= t_1^{-\lambda} \int_0^{\infty} (\bar{t} + t_1)^{\lambda-2k+n} e^{-\bar{t}-t_1} \bar{t}^\lambda d\bar{t} \\ &= t_1^{-\lambda} e^{-t_1} \int_0^{\infty} (t + t_1)^{\lambda-2k+n} e^{-t} t^\lambda dt. \end{aligned}$$

If $0 \leq t \leq t_1$, $t + t_1 \leq 2t_1$; if $t \geq t_1$, $t + t_1 \leq 2t$. Hence

$$\int_0^{t_1} (t + t_1)^{\lambda-2k+n} e^{-t} t^\lambda dt < (2t_1)^{\lambda-2k+n} \Gamma(\lambda + 1),$$

$$\int_{t_1}^{\infty} (t + t_1)^{\lambda-2k+n} e^{-t} t^\lambda dt < 2^{\lambda-2k+n} \Gamma(2\lambda - 2k + n + 1).$$

Consequently

$$I_2'' < t_1^{-\lambda} e^{-t_1} 2^{\lambda-2k+n} [t_1^{\lambda-2k+n} \Gamma(\lambda+1) + \Gamma(2\lambda-2k+n+1)],$$

$$(23) \quad |I_2| < \frac{\sec\alpha}{|z'|^n |\lambda|} \{\Gamma(\lambda-2k+n+1) + e^{-t_1} 2^{\lambda-2k+n} t_1^{-\lambda} [\Gamma(\lambda+1) t_1^{\lambda-2k+n} + \Gamma(2\lambda-2k+n+1)]\}.$$

For large values of $|z'| = |z|$, and for $\sec\alpha < C$, the sum of the upper bounds for $|I_1|$ and $|I_2|$ is of the order of $|z|^{-n}$; since this sum is independent of y, for $\sec\alpha < C$, the asymptotic expansion (5) holds also in a sector ($\sec\alpha < C$) which includes the negative half of the real axis.*

2. Suppose given the equation† and inequalities

$$(24) \quad \frac{d^2x}{dt^2} - \phi(t)x = 0,$$

$$\alpha^2 t^{m_1} < \phi(t) < \alpha^2 t^{m_2}, \qquad t > t_0, \quad m_1 > 0.$$

The auxiliary equation

$$(25) \quad \frac{d^2y}{dt^2} - \alpha^2 t^m y = 0, \qquad m \neq -2$$

can be transformed into

$$(26) \quad \frac{d^2\bar{y}}{dz^2} + \left[-\frac{1}{4} + \frac{\frac{1}{4} - p^2}{z^2}\right]\bar{y} = 0$$

by means of the substitutions

$$(27) \quad p = \frac{1}{m+2}, \qquad y = t^{-(m/4)}\bar{y}, \qquad z = \frac{4\alpha}{m+2} t^{(m+2)/2}.$$

Consequently any solution of (25) can be represented in the form

$$y = t^{-(m/4)} \left[C_1 W_{0,p}\left(\frac{4\alpha}{m+2} t^{(m+2)/2}\right) + C_2 W_{0,p}\left(\frac{-4\alpha}{m+2} t^{(m+2)/2}\right)\right].$$

---

* From the preceding developments we obtain

$$W_{k,m}(z) = e^{-(z/2)} z^k \left\{ a_1 + \frac{a_2}{z} + \cdots + \frac{a_{n+1}}{z^{n+1}} + R_{n+1}(z) \right\},$$

where $|R_{n+1}(z)| < A_{n+1}|z|^{-(n+1)}$. Consequently

$$W_{k,m}(z) = e^{-(z/2)} z^k \left\{ a_1 + \cdots + \frac{a_n}{z^n} + \bar{R}_n(z) \right\}$$

$$|\bar{R}_n(z)| = \left| \frac{a_{n+1}}{z^{n+1}} + R_{n+1}(z) \right| \leq \frac{|a_{n+1}| + A_{n+1}}{|z^{n+1}|}.$$

Hence it can be assumed that $\bar{R}_n(z)$ is of the order of $|z|^{-n-1}$.

† On certain linear differential equations of the second order, *Annals of Math.*, Vol. 24, No. 1, 1922.

Since $-1 < -\frac{1}{2} + p < 0$, the results of the first section can be applied. Also, from (1) it is easily shown that $W_{-0, m}(-z)$ is equal to a real function plus a solution of the order of $e^{-(z/2)}$ under the hypotheses of section 1. Hence the asymptotic expansion for $W_{0,p}(-z)$ is that of a real solution of (26).

Consequently by an argument exactly similar to that employed at the end of § 2 of the paper referred to, it is seen that if $Y_2(t)$ is the solution of (24) passing through (t_0, x_0) which remains bounded for $t < t_0$, Y_2', Y_2'', the corresponding solutions of (25) for $m = m_1$, $m = m_2$ respectively, then for $t > t_0$,

$$(28)\qquad \begin{gathered} Y_2' < Y_2 < Y_2'', \\ Y_2^{(i)} = C_i t^{-(m_i/4)} e^{2\alpha/(m_i+2) t^{(m_i+2)/2}} \{1 + \overline{R}_1(t) t^{-(m_i+2)/2}\}. \end{gathered}$$

Also, if $m_1 < m_2 < -2$, $p_i < 0$, and z approaches zero as $t \to \infty$. Inequalities (28) hold again, with

$$\overline{Y}_2^{(i)} = C_i' t \{1 + \sum_{n=1}^{\infty} a_n t^{n(m_i+2)}\} + C_i''\{1 + \sum_{n=1}^{\infty} a_n' t^{n(m_i+2)/2}\},$$

as is seen by expressing the solutions of (26) in terms of the functions $M_{0,p}(z)$, $M_{0,-p}(z)$.*

* Whittaker and Watson, l.c.

SOME GEOMETRIC APPLICATIONS OF SYMMETRIC SUBSTITUTION GROUPS.

By Arnold Emch.

I. Introduction.

The geometry of the symmetric group in its fundamental aspects has been investigated in an important memoir by J. Veronese.* Meanwhile the literature on invariant forms under finite collineation groups has become quite extensive.

The n letters $a_1, a_2, a_3, \cdots, a_n$ of a substitution may be considered as the homogeneous coördinates of a point of a projective $(n-1)$-space, so that to the $n!$ substitutions of the group correspond the same number of points which lie on the hyper-quadric

$$(1)\qquad \sum_{i=1}^{n}(x_i^2)\cdot\sum_{\substack{i,k=1\\ i\neq k}}^{n}(a_ia_k)-\sum_{\substack{i,k=1\\ i\neq k}}^{n}(x_ix_k)\cdot\sum_{i=1}^{n}(a_i^2)=0.$$

This may always be written in the form

$$(2)\qquad \lambda\left(\sum_{i=1}^{n}x_i\right)^2-\mu\sum_{\substack{i,k=1\\ i\neq k}}^{n}(x_i-x_k)^2=0.$$

For the purpose of this paper I shall quote in substance a few theorems for $n=3$ and $n=4$; i.e., for the groups G_6 and G_{24}.

Theorem 1: *All sextuples of points of the G_6 lie on a pencil of conics, which touch the lines*

$$x_1+\epsilon x_2+\epsilon^2x_3=0$$

and

$$x_1+\epsilon^2x_2+\epsilon x_3=0$$

at their intersections I and J with the unit-line $e\equiv x_1+x_2+x_3=0$.

Denoting by $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ the intersections of the sides $x_1=0$, $x_2=0$, $x_3=0$ with e, by $A_1A_2A_3$ and E the coördinate-triangle and unit-point, we have

Theorem 2: *The points of every sextuple determine 3 involutions on the corresponding conic, with $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ as centers and $A_1E_1=l_1$, $A_2E_2=l_2$, $A_3E_3=l_3$ respectively as axes of the involution. If these cut e in the same*

* "Interprétation géométrique de la théorie des substitutions de n letters, particulièrement pour $n=3, 4, 5, 6$, en relation avec les groupes de l'Hexagramme mystique," *Annali di Matematica*, Vol. XI, Ser. II, pp. 93–236 (July, 1882).

order in $\mathfrak{C}_1'$, $\mathfrak{C}_2'$, $\mathfrak{C}_3'$, *then* $\mathfrak{C}_1\mathfrak{C}_1'$, $\mathfrak{C}_2\mathfrak{C}_2'$, $\mathfrak{C}_3\mathfrak{C}_3'$ *are pairs of an involution with* I *and* J *as double points.*

THEOREM 3: *There exists a group of collineations, simply isomorphic with the* G_6, *which leaves all conics of the pencil of the group invariant, and which permutes three associated involutions of a sextuple on every conic of the group.*

For the G_{24} we have

THEOREM 4: *Any set of* 24 *points of the* G_{24} *lies on* 16 *conics of a quadric whose planes by four pass through the four lines* s_i *cut out from the unit-plane* e *by the coördinate-planes* $x_i = 0$. *The points of the group lie two by two on* 72 *lines which in sets of* 12 *pass through the six vertices* $\mathfrak{C}_{ik}$ *of the quadrilateral* $s_1s_2s_3s_4$. 24 *points of the* S_{24} *form* 6 *involutions with the* $\mathfrak{C}_{ik}$*'s as centers and the* 6 *planes through the edges of the coördinate-tetrahedron and the unit-point, taken in the proper order, as axial planes. All quadrics of the* G_{24} *form a pencil and touch each other and the cone* $\Sigma(x_i - x_k)^2 = 0$ *along its intersection with* e.

$\mathfrak{C}_{ik}$ is the intersection of $\overline{A_iA_k}$ with e. These and related theorems may be immediately generalized for the $G_{n!}$, but nothing essentially new would be gained by doing so.

It is the purpose of this paper to study some of the curves and surfaces which are associated with these groups, i.e., the G_6 and G_{24}, and are invariant in the isomorphic groups of collineation.

Denoting a substitution of the collineation group G_6 by S_{ikl}, we have

$$S_{ikl} \equiv \begin{cases} \rho x_1' = x_i \\ \rho x_2' = x_k \\ \rho x_3' = x_l \end{cases} \quad \begin{matrix} (i, k, l = 1, 2, 3) \\ (i \neq k \neq l). \end{matrix} \tag{3}$$

For the G_{24} we have

$$S_{ijkl} \equiv \begin{cases} \rho x_1' = x_i \\ \rho x_2' = x_j \\ \rho x_3' = x_k \\ \rho x_4' = x_l \end{cases} \quad \begin{matrix} (i, j, k, l = 1, 2, 3, 4) \\ (i \neq j \neq k \neq l). \end{matrix} \tag{4}$$

II. INVARIANT PLANE n-ICS OF THE G_6.

§ 1. *General Case.*

1. Let $\varphi_1 = \Sigma x_i$, $\varphi_2 = \Sigma x_ix_k$, $\varphi_3 = x_1x_2x_3$ denote the elementary symmetric ternary forms, then every symmetric ternary n-ic, or curve of order n, of this type may be written in the form

$$\begin{aligned} C_n \equiv \lambda_0\varphi_1^n + \lambda_1\varphi_1^{n-2}\varphi_2 + \lambda_2\varphi_1^{n-3}\varphi_3 + \lambda_3\varphi_1^{n-4}\varphi_2^2 \\ + \lambda_4\varphi_1^{n-5}\varphi_2\varphi_3 + \lambda_5\varphi_1^{n-6}\varphi_2^3 + \lambda_6\varphi_1^{n-6}\varphi_3^2 + \cdots = 0. \end{aligned} \tag{5}$$

Such an n-ic is obviously invariant under the group of collineations G_6 represented by (3) and contains ∞^1 sextuples S_6 of points of the G_6.

Two n-ics of this kind intersect in n^2 points which group themselves into a finite number $\mu \leqq n/6$ of sextuples, while the rest of the common points, $n^2 - \mu$, are absorbed in a definite manner by $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J as will be shown in the cases of cubics, quartics, quintics, and sextics.

The determination of the exact number of effective constants for the n-ic (5) is a well-known problem of partition in number theory.* If the exponents of φ_1, φ_2, φ_3 in the general term of (5) are α, β, γ, there is

$$\alpha + 2\beta + 3\gamma = n. \tag{6}$$

There are as many distinct terms in (5) as there are positive integral solutions α, β, γ, for a given n, of the diophantine equation (6).

If this number is N, the number of effective constants is $N - 1$. N is the largest positive integer which is equal or comes nearest to $(n + 3)^2/12$. As the six points of an S_6 determine three involutions on the conic K_6 associated with the S_6, with $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ as centers and $x_2 - x_3 = 0$, $x_3 - x_1 = 0$, $x_1 - x_2 = 0$ as axes of involution, to a point P on K_6 corresponds a point Q of S_6, so that $\overline{PQ}$ passes through $\mathfrak{E}_1$ and cuts $x_2 - x_3 = 0$ in a point R so that $(\mathfrak{E}_1 RPQ) = -1$. When P approaches a point R on the line $x_2 - x_3 = 0$, Q does the same thing. The same situation exists for the other involutions. Hence any conic K_6 of the S_6 cuts the lines $l_1 \equiv x_2 - x_3 = 0$, $l_2 \equiv x_3 - x_1 = 0$, $l_3 \equiv x_1 - x_2 = 0$ in six points so that the tangents to K_6 at these points, in pairs, pass through $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$. This property may be extended to any general invariant n-ic of the S_6, as will appear from the polar of $\mathfrak{E}_i$ later on, so that we have

THEOREM 5: *A general invariant n-ic C_n, $F(x_1, x_2, x_3) = 0$ cuts each of the three lines $x_i - x_k = 0$ in n points, so that the tangents to the n-ic at these points pass through the corresponding point $\mathfrak{E}_i$.*

If a point P_{123} lies on $x_1 - x_2 = 0$, the six points of the S_6 coincide by twos, i.e., $P_{123} = P_{213} = P_{113}$, $P_{132} = P_{231} = P_{131}$, $P_{312} = P_{321} = P_{311}$, and their joins pass, in the limit, through $\mathfrak{E}_3$, $\mathfrak{E}_2$, $\mathfrak{E}_1$. Hence the

THEOREM 6: *If an invariant conic K_6 is tangent to an invariant C_n at one of the points of intersection of the C_n with the lines l_1, l_2, l_3, say l_1, then K_6 touches the C_n in two other points which lie on l_2 and l_3. There are, in general, n such conics.*

That the C_n cuts, say l_1, in n points with the tangents at these points passing through $\mathfrak{E}_1$ is corroborated by the fact that the first polar of $\mathfrak{E}_1$ with respect to the C_n breaks up into the line l_1 and an $(n - 2)$-ic.

When a conic K_6 touches the C_n in a point which does not lie on a line

* See Dickson, "History of the Theory of Numbers," Vol. II, Chap. III, pp. 101–164.

l, then it touches the C_n in 5 other points. The six points of tangency of the K_6 and C_n form, of course, a sextuple of the S_6.

To determine the number of such conics it must be remembered that all K_6's of the S_6 form a pencil

$$\varphi_1^2 + \lambda\varphi_2 = 0.$$

Such a conic is tangent to the n-ic $F = 0$, when

$$2\varphi_1 + \lambda\frac{\partial\varphi_2}{\partial x_i} = \mu\frac{\partial F}{\partial x_i}, \qquad i = 1, 2, 3.$$

Now $\partial\varphi_2/\partial x_i = \varphi_1 - x_i$. Denoting $\partial F/\partial x_i$ by F_i, this leads to the condition

$$\begin{vmatrix} 2\varphi_1 & \varphi_1 - x_1 & F_1 \\ 2\varphi_1 & \varphi_1 - x_2 & F_2 \\ 2\varphi_1 & \varphi_1 - x_3 & F_3 \end{vmatrix} = 0,$$

or

$$x_1(F_2 - F_3) + x_2(F_3 - F_1) + x_3(F_1 - F_2) = 0.$$

Computing this by the use of (5), it reduces to the form

$$(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)\Phi^{n-3}(x_1, x_2, x_3) = 0.$$

This is a degenerate n-ic which cuts the C_n in n^2 points which are points of tangency of conics of the group. Obviously, among these points are included the $3n$ points cut out by the C_n on the lines l. Outside of these points there are $n^2 - 3n$ points of tangency of conics with the C_n. In case that the C_n and the Φ^{n-3} have no points in common which lie on the unit-line, $n^2 - 3n$ is the number of tangencies of proper conics, and this number is a multiple of 6; there are then $\frac{1}{6}(n^2 - 3n)$ such conics touching the C_n in six points. If k points of intersection of F and Φ^{n-3} lie on the unit-line, the number of conics is reduced accordingly.

Summing up we may state the

THEOREM 7: *If k points of intersections of C_n and Φ^{n-3} are absorbed by points on the unit-line, then there are, in general, n invariant tri-tangent conics, with points of tangency on the lines $l_1l_2l_3$, and $\frac{1}{6}(n^2 - 3n - k)$ hexatangent invariant conics of the C_n.*

In case of a cubic there are merely 3 tritangent conics. For a sextic there are 6 tritangent conics and 3 hexatangent conics.

2. **Double Tangents and Double Points.**—The polar of $\mathfrak{E}_1$ with respect to the C_n is simply $F_2 - F_3 = 0$, or

$$(x_2 - x_3)\{\lambda_1\varphi_1^{n-2} + 2\lambda_3\varphi_1^{n-4}\varphi_2 + \lambda_4\varphi_1^{n-5}\varphi_3 + 3\lambda_5\varphi_1^{n-6}\varphi_2^2 \\ + \cdots + x_1(\lambda_2\varphi_1^{n-3} + \lambda_4\varphi_1^{n-5}\varphi_2 + 2\lambda_6\varphi_1^{n-6}\varphi_3 + \cdots)\} = 0,$$

or

$$(x_2 - x_3)\psi_{n-2} = 0.$$

Hence, there are in general $n(n-2)$ tangents from $\mathfrak{E}_1$ to the C_n whose

points of tangency P do not lie on the line $x_2 - x_3 = 0$. Suppose $\mathfrak{E}_1 P$ as such a tangent. On account of the invariance of C_n, there is another point Q on C_n and l_1 which is a point of tangency.

If the C_n and ψ_{n-2} have intersections on the unit-line l, the number of tangents from the points $\mathfrak{E}_i$ is reduced by a certain number k. As every proper tangent from $\mathfrak{E}_i$ to the C_n touches C_n in another point, we may state the

THEOREM 8: *From every point $\mathfrak{E}_i$ there are $\frac{1}{2}\{n(n-2)-k\}$ double tangents to the C_n. There are altogether $\frac{3}{2}\{n(n-2)-k\}$ such double tangents.*

If an invariant n-ic has a multiple point at $\mathfrak{E}_1$, then it also has multiple points of the same type at $\mathfrak{E}_2$ and $\mathfrak{E}_3$. Likewise, multiplicities of the same sort appear simultaneously at A_1, A_2, A_3, and at I and J. According to theorem 5 a multiplicity in a point of the axes l_i of involutory perspective occurs when two branches come into contact at such a point. This may be an ordinary double point or a tacnode. If an n-ic passes through E it will have a singularity at E. Outside of these points the multiple points of an n-ic, if there are any, lie by groups of sextuples on conics of the G_6.

That E is a singularity for an invariant n-ic passing through E can easily be proved as follows: Suppose that b is a simple branch of such a curve through E and in its neighborhood whose tangent at E does not coincide with one of the l_i's. When a point P describes b the 5 equivalent points of P of the sextuple describe 5 other branches of the same curve through E. Hence E is a sextuple point of the n-ic. When b has one of the l_i's as a tangent at E, then as l_i is invariant in one collineation of the G_6, there will be a second branch of the curve having the same l_i as a tangent. The same situation exists for the other two l_i's. Hence the sextuple point may become a triple tacnode. Imposing the single condition that the curve shall have a double point and no higher singularity at E leads to the result that in such a case E is an isolated double point with $\overline{EI}$ and $\overline{EJ}$ as imaginary tangents. In fact this is verified by making C_n pass through E. This establishes a linear relation between all λ's in (5). When this obtains, $\partial C_n/\partial x_1$, $\partial C_n/\partial x_2$, $\partial C_n/\partial x_3$ vanish at E, which establishes the fact that E is a singularity. To ascertain its nature, we may take A_2EA_3 as the new coordinate-triangle, so that $x_1 = x_1'$, $x_2 = x_1' - x_2'$, $x_3 = x_1' - x_3'$. Substituting these in (5), the coefficients of $x_1'^n$, $x_1'^{n-1}$ vanish identically, and the coefficient of $x_1'^{n-2}$ contains $x_2'^2 - x_2'x_3' + x_3'^2$ as a factor. This breaks up into two linear factors which are the transformed equations of $\overline{EI}$ and $\overline{EJ}$, and which represent the tangents to C_n at E. If ϵ represents a cube root of unity the equations of $\overline{EI}$ and $\overline{EJ}$ turn out to be

$$\overline{EI} \equiv x_1 + \epsilon x_2 + \epsilon^2 x_3 = 0,$$
$$\overline{EJ} \equiv x_1 + \epsilon^2 x_2 + \epsilon x_3 = 0.$$

Summing up we have

THEOREM 9: *Singularities of invariant n-ics occur simultaneously at the A's, $\mathfrak{E}$'s, I and J, if they occur at all at any of these sets of points. Singularities on the l_i's occur in triples belonging to the G_6. If the n-ic passes through E, then, in general, E is an isolated double point of the n-ic. For curves higher than the sixth order, E may become a sextuple or multiple sextuple point, including triple tacnodal sets. Outside of these, singularities, if they exist, occur in sextuples lying on conics of the G_6.*

It is of importance to know whether there exist rational invariant n-ics. The question can be answered affirmatively as follows: We may restrict ourselves to the cases where the double points occur in groups of sextuples only. For this purpose there must be

$$\delta = \frac{(n-1)(n-2)}{2} = 6m,$$

where m is a positive integer. Choosing $n = 12k + 5$, there is

$$\delta = 6(3k+1)(4k+1)$$

and

$$m = (3k+1)(4k+1).$$

m denotes the number of sextuples of double points. Clearly the lowest order for an irreducible rational n-ic is 5. The six double points of the rational invariant quintic lie on a conic of the G_6. The general quintic belonging to the G_6 passes singly through $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J.

That there are rational curves with other singularities than those in sextuples appears from the case of the sextic with double points at A_1, A_2, A_3, E and at the points of a given sextuple. Another case is the septimic with a sextuple point at E.

§ 2. *Cubic.*

The most general cubic of the S_6 may be written in the form

$$F = \varphi_1^3 + \lambda\varphi_1\varphi_2 + \mu\varphi_3 = 0. \tag{7}$$

It depends on two effective parameters λ, μ, and cuts the unit-line $e \equiv x_1 + x_2 + x_3 = 0$ in $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ which are points of inflexion of the cubic. The inflexional tangents at these points are

$$(\lambda+\mu)x_1 + \lambda x_2 + \lambda x_3 = 0,$$

$$\lambda x_1 + (\lambda+\mu)x_2 + \lambda x_3 = 0,$$

$$\lambda x_1 + \lambda x_2 + (\lambda+\mu)x_3 = 0.$$

If these are chosen as sides of a new coördinate triangle, F assumes the form

$$(x_1 + x_2 + x_3)^3 - \frac{(3\lambda + \mu)^3}{\lambda^3 + \lambda^2\mu - \mu^2} \cdot x_1x_2x_3 = 0. \tag{8}$$

Now it is known that every general (elliptic) cubic may be reduced to the form

$$x_1^3 + x_2^3 + x_3^3 - 3kx_1x_2x_3 = 0, \tag{9}$$

in which $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ are again inflexional points. If we choose the inflexional tangents again as sides of a new coördinate triangle, the general cubic assumes the form

$$(x_1 + x_2 + x_3)^3 - \frac{3(k + 2)^3}{k^2 + k + 1} \cdot x_1x_2x_3 = 0, \tag{10}$$

in which the parameter multiplying $x_1x_2x_3$ may have any value. But as, likewise, also the parameter (rational function of λ and μ) multiplying $x_1x_2x_3$ in the reduced form of the symmetric cubic may have any value, it is evident that the cubic invariant in the symmetric group S_6 is a general cubic.

There is however also a pencil of rational cubics in the G_6. If we make C_3 pass through E, by choosing $\mu = -27 - 9\lambda$, this pencil is

$$C_3 \equiv \varphi_1^3 - 27\varphi_3 + \lambda(\varphi_1\varphi_2 - 9\varphi_3) = 0. \tag{11}$$

§ 3. *Quartic.*

1. From § 1, 2, in case of a quartic,

$$\begin{aligned} C_4 &\equiv \lambda_0\varphi_1^4 + \lambda_1\varphi_1^2\varphi_2 + \lambda_2\varphi_1\varphi_3 + \lambda_3\varphi_2^2 = 0, \\ \psi_2 &= \lambda\varphi_1^2 + \lambda_2\varphi_1x_1 + 2\lambda_3\varphi_2 = 0. \end{aligned} \tag{12}$$

The two curves intersect in 8 points of which two are I and J on e $(k = 2)$. Outside of e there are therefore 6 points of intersection, and consequently 3 double tangents from every point $\mathfrak{E}_i$. The unit line e is obviously a double tangent to the C_4 at I and J. In this manner we have accounted for 10 double tangents of the C_4.

Every $\mathfrak{E}_i$ is the center of an involutory perspective collineation, with $x_j - x_k = 0$ as the axis of perspective (j and k being the indices chosen from 1, 2, 3, different from i), by which the C_4 is transformed into itself.

The Hessian of C_4 which is an invariant C_6 cuts the C_4 in 24 points of inflexion which lie in sextuples on four conics of the G_6.

Choosing IJE as the new coördinate triangle, i.e., putting

$$\begin{aligned} x_1 + \epsilon x_2 + \epsilon^2 x_3 &= \rho x_1', \\ x_1 + \epsilon^2 x^2 + \epsilon x_3 &= \rho x_2', \\ x_1 + x_2 + x_3 &= \rho x_3', \end{aligned}$$

the C_4 will assume the form

$$ax_3'^4 + bx_3'^2x_1'x_2' + cx_3'(x_1'^3 + x_2'^3) + dx_1'^2x_2'^2 = 0, \tag{13}$$

in which a, b, c, d are linear functions of λ_0, λ_1, λ_2, λ_3. This is precisely the form of the invariant quartic discussed by *Ciani*.*

The triply infinite linear system of C_4's contains, of course also, the pencil

$$\sum x_i^4 + 6\lambda \sum x_i^2x_k^2 = 0, \tag{14}$$

which is invariant under the 24 substitutions of the octahedral group

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ \pm x_i & \pm x_k & \pm x_j \end{pmatrix} \qquad i, k, j = 1, 2, 3, \quad i \neq k \neq j \neq i.$$

Among the curves of the pencil are two Kleinean curves, which are obtained for

$$\lambda = \frac{-1 + i\sqrt{7}}{4}, \qquad \lambda = \frac{-1 - i\sqrt{7}}{4}.$$

These cases are also discussed by Ciani, loc. cit.

2. **Double Tangents of Quartic.**—From the well-known fact that if α, β, ν, δ is a set of four of the 28 double tangents of a general quartic, whose points of contact lie on a conic φ, then the quartic has the form

$$\alpha\beta\nu\delta - \lambda\varphi^2 = 0, \tag{15}$$

it must be possible to put our general quartic in this form. Denoting by λ a parameter

$$C_4 = \lambda_0\varphi_1^4 + \lambda_1\varphi_1^2\varphi_2 + \lambda_2\varphi_1\varphi_3 + \lambda_3\varphi_2^2 = 0$$

may be written in the identical form

$$\varphi_1\{(\lambda_0\lambda^2 - \lambda_3)\varphi_1^3 + (\lambda_1\lambda^2 - 2\lambda_3\lambda)\varphi_1\varphi_2 + \lambda^2\lambda_2\varphi_3\} - \lambda_3(\varphi_1^2 + \lambda\varphi_2)^2 = 0.$$

Now $e \equiv \varphi_1$ is a double tangent, and from each $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ there are 3 double tangents. Sets of three out of these 9 may be chosen so that their product forms a symmetric cubic. Such a product is necessarily of the form

$$(\varphi_1 + \mu x_1)(\varphi_1 + \mu x_2)(\varphi_1 + \mu x_3) = (1 + \mu)\varphi_1^3 + \mu^2\varphi_1\varphi_2 + \mu^3\varphi_3.$$

Now it is possible to choose λ in such a manner that the cubic

$$(\lambda_0\lambda^2 - \lambda_3)\varphi_1^3 + (\lambda_1\lambda^2 - 2\lambda_3\lambda)\varphi_1\varphi_2 + \lambda^2\lambda_2\varphi_3 = 0$$

becomes reducible like the cubic with the parameter μ. For this purpose we must eliminate μ and ρ from the three equations

$$\lambda_0\lambda^2 - \lambda_3 = \rho(1 + \mu), \qquad \lambda_1\lambda^2 - 2\lambda_3\lambda = \rho\mu^2, \qquad \lambda_3\lambda^2 = \rho\mu^3.$$

* "I varii tipi possibili di quartiche piane piu volte omologico-armoniche," *Rendiconti del Circolo Matematico di Palermo*, Vol. XIII, pp. 347–373 (1899).

This leads to the cubic in λ

$$[\lambda_0\lambda_2^2 - \lambda_1^2(\lambda_1 + \lambda_2)]\lambda^3 + [6\lambda_1^2\lambda_3 + 4\lambda_1\lambda_2\lambda_3]\lambda^2 - [\lambda_2^2\lambda_3 + 8\lambda_1\lambda_3^2 + 4(\lambda_1 + \lambda_2)\lambda_3^2]\lambda + 8\lambda_3^3 = 0.$$

There are therefore three triples of double tangents. Hence

THEOREM 10: *The 9 double tangents through the $\mathfrak{E}_i$'s form three triples, whose 6 points of contact, together with I and J of course, lie on 3 conics of the G_6. The remaining 18 double tangents form 3 sextuples each circumscribed to a conic of the G_6. The twelve points of contact of each sextuple lie on two conics of the group.*

3. **The Quartic as an Envelope of Cubics.**—The conic $\varphi_1^2 + \lambda\varphi_2 = 0$ cuts the cubic

$$(\lambda_0\lambda^2 - \lambda_3)\varphi_1^3 + (\lambda_1\lambda^2 - 2\lambda_3\lambda)\varphi_1\varphi_2 + \lambda_2\lambda^2\varphi_3 = 0$$

in 6 points which are points of tangency of the quartic and the cubic. For every value of λ there is such a sextuple, so that the C_4 may be generated as the envelope of the system of cubics

$$(16) \qquad (\lambda_0\varphi_1^3 + \lambda_1\varphi_1\varphi_2 + \lambda_2\varphi_3)\lambda^2 - 2\lambda_3\varphi_1\varphi_2\lambda - \lambda_3\varphi_1^3 = 0.$$

In fact the discriminant of this cubic with respect to λ gives precisely the C_4. Through every point (x) there are evidently two cubics touching the C_4 along sextuples. When (x) is on the C_4, then the two cubics coincide.

THEOREM 11: *Every C_4 is enveloped by a system of invariant cubics of index 2. Through every sextuple S there are two cubics which touch the C_4 in points of sextuples. When S is on the C_4 the two tangent-cubics coincide.*

The other well-known systems of enveloping conics and cubics of the quartic do not belong to symmetric forms and shall therefore not be considered in this place.

§ 4. *Quintics.*

1. **System of Quintics and their Double Points.**—The general system of invariant quintics

$$(17) \qquad \lambda_0\varphi_1^5 + \lambda_1^3\varphi_1^3\varphi_2 + \lambda_2\varphi_1^2\varphi_3 + \lambda_3\varphi_1\varphi_2^2 + \lambda_4\varphi_2\varphi_3 = 0$$

depends on four effective constants. All quintics of the system pass through the five fixed points $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J and have $\overline{EI}$ and $\overline{EJ}$ as common tangents at I and J. Two quintics intersect in 25 points, of which 7 are absorbed by the five fixed points. The remaining 18 intersections form three sextuples of the G_6. Four independent sextuples determine a quintic uniquely and the quintics through three fixed sextuples form a pencil.

As a double point absorbs three conditions, any point P in a general position may be taken as a double point of a quintic, so that also the five equivalent points of the group are double points.

THEOREM 12: *The points of a sextuple are therefore the double points of a rational quintic.*

There can be only one quintic with a given sextuple of double points, since two distinct quintics with these as common double points would intersect in 27 points, which is impossible.

The quintics of the set

$$\lambda_2\varphi_1^2\varphi_3 + \lambda_3\varphi_1\varphi_2^2 + \lambda_4\varphi_2\varphi_3 = 0 \tag{18}$$

have A_1, A_2, A_3 as double points. When $\lambda_4 = -3\lambda_2 - 9\lambda_3$, then also E becomes a double point (isolated).

2. **Quintics as Envelopes and Problems of Closure.**—Multiplying the quintic by φ_1 we obtain the reducible sextic

$$\lambda_0\varphi_1^6 + \lambda_1\varphi_1^4\varphi_2 + \lambda_2\varphi_1^3\varphi_3 + \lambda_3\varphi_1^2\varphi_2^2 + \lambda_4\varphi_1\varphi_2\varphi_3 = 0, \tag{19}$$

which by transformation

$$\rho y_1 = \varphi_1^3, \qquad \rho y_2 = \varphi_1\varphi_2, \qquad \rho y_3 = \varphi_3 \tag{20}$$

is mapped on the conic K

$$\lambda_0 y_1^2 + \lambda_1 y_1 y_2 + \lambda_2 y_1 y_3 + \lambda_3 y_2^2 + \lambda_4 y_2 y_3 = 0 \tag{21}$$

in the (y)-plane. To the line $\varphi_1 = 0$ in the (x)-plane corresponds the point $(0, 0, 1)$ on the conic. Obviously the points $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, I, J of the quintic are mapped into $(0, 0, 1)$.

Conversely to a line $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0$ corresponds the cubic

$$\alpha_1\varphi_1^3 + \alpha_2\varphi_1\varphi_2 + \alpha_3\varphi_3 = 0. \tag{22}$$

To a conic in (y) corresponds a sextic in (x), and so forth. To a tangent t of (21) corresponds a cubic C_3 which touches the quintic along the points of a sextuple which absorb 12 of the 15 points of intersection of the cubic and the quintic. The remaining three points of intersection lie at $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$. As the conic (21) is enveloped by its system of tangents we have

THEOREM 13: *A given general quintic of G_6 is enveloped by a definite system of ∞^1 cubics belonging to G_6, so that through every sextuple there are, in general, two sixfold tangent cubics.*

The ∞^2 double tangent conics of K may be written in the form

$$(\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3)^2 + K = 0. \tag{23}$$

From the algebraic form of (23) it is easily seen that through two fixed points in (y) there are in general four double tangent conics to K. Hence, when we consider the ∞^1 double tangent conics (23) through a fixed point we obtain a system such that through every point there are four conics of the system.

Transforming back to the (x)-plane we have

THEOREM 14: *Every point in (y) determines a system of sextics which envelopes the quintic such that every sextic touches the quintic along the points of two sextuples. Through every sextuple in a general position there are four such tangent sextics.*

These sextics have the form

$$(\alpha_1\varphi_1^3 + \alpha_2\varphi_1\varphi_2 + \alpha_3\varphi_3)^2 + \varphi_1 C_5 = 0, \tag{24}$$

where C_5 denotes the quintic. In the intersection of the sextic and the quintic $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ count for two points each, so that the complete intersection consists of $2\cdot 2\cdot 6 + 2\cdot 3 = 30$ points.

It is evident that other projective properties of the conic have their equivalent on the quintic. For example, consider two conics K, say K_1 and K_2, with a Poncelet polygone of n sides inscribed in K_1 and circumscribed to K_2. Going back to the (x)-plane we have

THEOREM 15: *Given two quintics $C_5^{(1)}$, $C_5^{(2)}$ of the G_6. Through any sextuple S_0 of $C_5^{(1)}$ pass a cubic touching $C_5^{(2)}$ along the points of a sextuple and cutting $C_5^{(1)}$ in a second sextuple S_1. Through S_1 pass another cubic touching C_5^2 in the same manner and cutting C_5^1 in a third sextuple S_2; suppose that after continuing this process n times, S_n coincides with S_0. If this happens once, S_n will always coincide with S_0, no matter what sextuple S_0 we choose on C_5^1.*

§ 5. *Sextics.*

1. **Systems of Sextics and Double Points.**—A general sextic of the G_6:

$$\lambda_0\varphi_1^6 + \lambda_1\varphi_1^4\varphi_2 + \lambda_2\varphi_1^3\varphi_3 + \lambda_3\varphi_1^2\varphi_2^2 + \lambda_4\varphi_1\varphi_2\varphi_3 + \lambda_5\varphi_2^3 + \lambda_6\varphi_3^2 = 0, \tag{25}$$

depends on six effective constants, so that six independent sextuples determine a sextic completely. Five such sextuples determine a pencil. From this follows

THEOREM 16: *All sextics which pass through five independent fixed sextuples pass through a sixth fixed sextuple.*

The points of a sextuple are double points of all sextics of a definite system. In addition to such a sextuple of double points, a sextic may have double points at A_1, A_2, A_3 and E, so that the sextic becomes a rational sextic. There is just one sextic with these double points since two sextics with the same double points would intersect in 40 points. The sextic

$$\lambda_4\varphi_1\varphi_2\varphi_3 + \lambda_5\varphi_2^3 + \lambda_6\varphi_3^2 = 0 \tag{26}$$

has triple points at A_1, A_2, A_3. Moreover if we choose $\lambda_6 = -9\lambda_4 - 27\lambda_5$, we obtain a pencil of rational sextics

$$\varphi_1\varphi_2\varphi_3 - 9\varphi_3^2 - \lambda(27\varphi_3^2 - \varphi_2^3) = 0, \tag{27}$$

which at A_1, A_2, A_3 have the common tangents $x_2 + x_3 = 0$, $x_3 + x_1 = 0$, $x_1 + x_2 = 0$, and at E the common tangents $\overline{EI}$ and $\overline{EJ}$.* Of the 36 points of intersection of two sextics 10 are absorbed by each A_1, A_2, A_3 (9 on account of the triple point, 1 on account of the common tangents at each point); 6 by E (double points with common tangents).

2. **Sextics as Envelopes.**—To the reducible nonic consisting of the product of φ_1^3 and the sextic corresponds in the (y)-plane, by the transformation (20), the cubic

$$(28)\quad \lambda_0 y_1^3 + \lambda_1 y_1^2 y_2 + \lambda_2 y_1^2 y_3 + \lambda_3 y_1 y_2^2 + \lambda_4 y_1 y_2 y_3 + \lambda_5 y_2^3 + \lambda_6 y_1 y_3^2 = 0,$$

which, in general, is elliptic. To the line $\varphi_1 = 0$ corresponds the point $(y) = (0, 0, 1)$, to the factor φ_1^3 of the nonic this point three times. To the intersections of the sextic with $\varphi_1 = 0$ corresponds this same point. To every sextuple of the sextic corresponds a point of the cubic (28). Conversely to every point of the cubic corresponds a sextuple of the sextic. More generally, to every sextuple in (x) corresponds a point in (y), and conversely. To lines, conics, etc., in (y) correspond symmetric cubics, sextics, etc., in (x). Hence, to the geometry of points, lines, conics, . . . in (y) corresponds abstractly the same geometry of sextuples, cubics, sextics, . . . of the G_6 in (x).

By means of this correspondence we are able to state immediately a number of theorems in the (x)-plane which are the equivalents of those in the (y)-plane. We shall restrict ourselves to some of the most important.

An elliptic cubic has 9 inflexions which lie 3 by 3 on 12 lines. In the (x)-plane we have

THEOREM 17: *There are 9 cubics which osculate a given (general) sextic of the G_6 in points of a sextuple. The nine sextuples of osculating points lie 3 by 3 on 12 cubics.*

Again an elliptic cubic admits of 27 conics with sextactic contact; hence

THEOREM 18: *There are 27 sextics which touch a given sextic in sextuples of sextactic points.*

With every inflexion of an elliptic cubic is associated a system of ∞^2 tritangent conics. Through every point in a general position there is a system of ∞^1 such conics, which envelope the cubic. In the (x)-plane we have accordingly

THEOREM 19: *A given sextic may be generated in nine ways by ∞^2 systems of enveloping sextics. Every enveloping sextic touches the given sextic in the points of three sextuples.*

* For other special types of sextics invariant under the G_6, for example the G_{360}, see A. B. Coble, "An Invariant Condition for Certain Automorphic Algebraic Forms." AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII, pp. 333–366 (1906).

There are 9 systems of ∞^1 doubly osculating conics for a given cubic. Thus

THEOREM 20: *Every sextic admits of 9 systems of ∞^1 doubly osculating sextics. Every enveloping sextic osculates the given sextic in the points of two sextuples.*

3. **Problems of Closure.**—Let P and Q be two points on the cubic C_3. Through P draw any line l cutting C_3 in A_l and B_l. Let the join B_lQ cut C_3 in a third point C_l; let C_lP cut C_3 in D_l; finally, let D_lQ cut C_3 in E_l. If E_l coincides with A_l, then this coincidence will take place, no matter what initial line l we draw through P. We have accordingly

THEOREM 21: *Let P and Q be two sextuples on the sextic C_6. Through P draw any cubic l cutting C_6 in the sextuples A_l and B_l. Let the cubic through B_l and Q cut C_6 in a third sextuple C_l; let the cubic through C_l and P cut C_6 in the sextuple D_l; finally let the cubic through D_l and Q cut C_6 in the sextuple E_l. If E_l coincides with A_l, then this coincidence will take place, no matter what initial cubic l we pass through P.*

Other equivalent theorems might be stated with equal ease.

III. INVARIANT SURFACES AND CURVES OF THE G_{24}.

§ 1. *The General Invariant n-ic.*

Denoting the elementary symmetric functions in the quaternary field again by

$$\varphi_1 = \Sigma x_i, \qquad \varphi_2 = \Sigma x_i x_k, \qquad \varphi_3 = \Sigma x_i x_j x_k, \qquad \varphi_4 = x_1 x_2 x_3 x_4,$$

the general n-ic may be written in the form

$$\sum_{i=0}^{N-1} \lambda_i \varphi_1^\alpha \varphi_2^\beta \varphi_3^\gamma \varphi_4^\delta = 0, \tag{29}$$

$$\alpha + 2\beta + 3\gamma + 4\delta = n, \tag{30}$$

so that the number N of effective constants is equal to the positive integral solution of this diophantine equation, diminished by one. It is not difficult to find the number N for a given numerical integral value of n. For example, the systems of quadrics, cubics, quartics, quintics, sextics depend on 1, 2, 4, 5, 8 effective constants.

In space of $m-1$ dimensions the symmetric n-ic $(n \geqq m)$

$$\sum_{i=0}^{N-1} \varphi_1^\alpha \varphi_2^\beta \varphi_3^\nu \cdots \varphi_m^\mu = 0 \tag{31}$$

depends on N effective constants, whose number depends analogously on the partition problem in number theory.

$$\alpha + 2\beta + 3\nu + \cdots + m\mu = n. \tag{32}$$

In what follows I shall restrict myself to a short discussion of cubics and the sextic curves obtained as intersections of quadrics and cubics of the G_{24}.

§ 2. *The 27 Lines on a Symmetric Cubic.*

The symmetric cubic C_3

$$\varphi_1^3 + \lambda\varphi_1\varphi_2 + \mu\varphi_3 = 0, \tag{33}$$

or

$$\begin{aligned}&(x_1 + x_2 + x_3 + x_4)^3\\ &+ \lambda(x_1 + x_2 + x_3 + x_4)[(x_1 + x_2)(x_3 + x_4) + x_1x_2 + x_3x_4]\\ &\qquad + \mu[(x_1 + x_2)x_3x_4 + (x_3 + x_4)x_1x_2] = 0\end{aligned} \tag{34}$$

is satisfied by any point of the three lines

$$l_1\begin{cases}x_1 + x_2 = 0\\ x_3 + x_4 = 0\end{cases}\qquad l_2\begin{cases}x_1 + x_3 = 0\\ x_2 + x_4 = 0\end{cases}\qquad l_3\begin{cases}x_1 + x_4 = 0\\ x_2 + x_3 = 0.\end{cases}$$

These lines lie on the unit-plane $x_1 + x_2 + x_3 + x_4 = 0$ and form one of the 45 triangles of the cubic. To find the remaining 24 lines, pass any plane $x_3 + x_4 = \theta(x_1 + x_2)$ through l_1. This will cut C_3 in a conic whose projection upon the $(x_1x_2x_3)$-plane is obtained by the elimination of x_4. There is $x_4 = \theta(x_1 + x_2) - x_3$, so that (34) becomes, after dividing through by $(x_1 + x_2)$ and rearranging,

$$\begin{aligned}&[(1+\theta)^3 + \lambda(1+\theta)\theta]x_1^2 + [(1+\theta)^3 + \lambda(1+\theta)\theta]x_2^2 - [\lambda(1+\theta)+\mu]x_3^2\\ &\qquad + 2\left[(1+\theta)^3 + \lambda(1+\theta)\theta + \frac{\lambda}{2}(1+\theta) + \frac{\mu}{2}\theta\right]x_1x_2\\ &\qquad + 2\left[\frac{\lambda}{2}(1+\theta)\theta + \frac{\mu}{2}\theta\right]x_1x_3 + 2\left[\frac{\lambda}{2}(1+\theta)\theta + \frac{\mu}{2}\theta\right]x_2x_3 = 0.\end{aligned}$$

This conic degenerates into two lines, when the discriminant

$$\begin{vmatrix}[(1+\theta)^3+\lambda(1+\theta)\theta] & \left[(1+\theta)^3+\lambda(1+\theta)\theta+\frac{\lambda}{2}(1+\theta)+\frac{\mu}{2}\theta\right] & \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right]\\ \left[(1+\theta)^3+\lambda(1+\theta)\theta+\frac{\lambda}{2}(1+\theta)+\frac{\mu}{2}\theta\right] & [(1+\theta)^3+\lambda(1+\theta)\theta] & \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right]\\ \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right] & \left[\frac{\lambda}{2}(1+\theta)\theta+\frac{\mu}{2}\theta\right] & -[\lambda(1+\theta)+\mu]\end{vmatrix} = 0.$$

Subtracting the second from the first line and factoring we get

$$\begin{aligned}&\tfrac{1}{2}[\lambda(1+\theta)+\mu\theta]\cdot\\ &[\lambda(1+\theta)+\mu]\cdot\end{aligned}\begin{vmatrix}-1 & 1 & 0\\ A & B & C\\ \dfrac{\theta}{2} & \dfrac{\theta}{2} & -1\end{vmatrix} = 0$$

or

$$(1+\theta)[\lambda(1+\theta)+\mu\theta][\lambda(1+\theta)+\mu][4(1+\theta)^2+4\lambda\theta+\lambda+\mu\theta+\lambda\theta^2]=0.$$

The last factor may be written

$$(4+\lambda)\theta^2+(8+4\lambda+\mu)\theta+4+\lambda.$$

This equated to zero gives for the roots

$$\theta_{4,5}=\frac{-8-4\lambda-\mu\pm\sqrt{(8+4\lambda+\mu)^2-4(4+\lambda)^2}}{8+2\lambda}.$$

There are, therefore, 5 planes through l_1 which cut C_3 in pairs of lines. The parameters of these planes are

$$\theta_1=-1,\qquad \theta_2=-\frac{\lambda}{\lambda+\mu},\qquad \theta_3=-\frac{\lambda+\mu}{\lambda},$$

$$\theta_4=\frac{-8-4\lambda-\mu+\sqrt{(8+4\lambda+\mu)^2-(8+2\lambda)^2}}{8+2\lambda},$$

$$\theta_5=\frac{8+2\lambda}{-8-4\lambda-\mu+\sqrt{(8+4\lambda+\mu)^2-(8+2\lambda)^2}}.$$

The plane $\theta_1=-1$, of course, cuts C_3 in l_2 and l_3. As θ_2 and θ_3, as well as θ_4 and θ_5, are reciprocal to each other, the planes θ_2 and θ_3 are permuted by the substitutions $\binom{1234}{3412}$; $\binom{1234}{4312}$; $\binom{1234}{4321}$; $\binom{1234}{3412}$. The same substitutions permute θ_4 and θ_5. The same equivalent collineations transform θ_2 into θ_3, and θ_4 into θ_5. On the other hand, the substitutions (collineations)

$$\binom{1234}{1243}\quad\binom{1234}{2134}\quad\binom{1234}{2143}$$

leave each of those planes invariant.

On account of the symmetric character of the equation of C_3, precisely the same parameters and substitutions for the planes through l_2 and l_3, cutting C_3 in pairs of lines, are obtained. Hence

THEOREM 22: *The effective determination of the 27 lines of a symmetric cubic is possible by the solution of three linear and one quadratic equation with a parameter θ as the unknown.*

The three lines l_1, l_2, l_3 are left invariant or are permuted by the 24 substitutions of the symmetric group. Every substitution which permutes l_i and l_k $(i, k=1, 2, 3)$ also permutes the planes through l_i and l_k with the same parameters.

§ 3. *Sextic Curves of the G_{24}.*

Two cubics

$$F=a\varphi_1^3+b\varphi_1\varphi_2+c\varphi_3=0, \tag{35}$$

$$G=d\varphi_1^3+e\varphi_1\varphi_2+f\varphi_3=0 \tag{36}$$

of the set of symmetric quaternary cubics intersect in a space curve of order 9 which degenerates into a sextic S and three fixed lines C_3 ($\varphi_1 = 0$, $\varphi_3 = 0$) in the unit-plane.

$$(37) \qquad Q^* = \frac{fF - cG}{\varphi_1} = (af - cd)\varphi_1^2 - (ce - bf)\varphi_2 = 0$$

is a quadric through S. Conversely when a definite quadric

$$(38) \qquad Q = \varphi_2 - \lambda\varphi_1^2 = 0$$

is given, any cubic through the sextic on Q and F may be written in the form

$$(39) \qquad (a + \lambda b)\varphi_1^3 + c\varphi_3 = 0.$$

From this follows that all possible sextics of the group on the quadric Q are cut out by a pencil of cubics which osculate along the three lines C_3.

On every sextic there is a simply infinite set of 24 points of the group which lie two by two on 12 lines through each of the $\mathfrak{E}_{ik}$'s. Hence, from each of these points the sextic is projected upon a plane into a curve all of whose points are double points, hence into a cubic. From this follows

THEOREM 23: *The sextics of the group are of genus* 4 *and form a set. Every sextic of the group lies on* 6 *cubic cones with their vertices at the* $\mathfrak{E}_{ik}$'s.

It is not difficult to find the equations of these cones. For example if we write $Q = \varphi_1^2 + \lambda\varphi_2$, $F = \varphi_1^3 + \mu\varphi_1\varphi_2 + \nu\varphi_3$, the cone with $\mathfrak{E}_{34}$ as a vertex has the form

$$(40) \quad \lambda(\varphi_1^3 + \mu\varphi_1\varphi_2 + \nu\varphi_3) - (\varphi_1^2 + \lambda\varphi_2)\{\mu(x_1 + x_2) + (\mu + \nu)(x_3 + x_4)\} = 0.$$

The relation between the six cones may be stated in

THEOREM 24: *The cubic cone through the sextic, with* $\mathfrak{E}_{ij}$ *as a vertex, osculates the plane* $x_i + x_j - x_k - x_l = 0$ *along the line* $\mathfrak{E}_{ij}\mathfrak{E}_{kl}$. *Two cubic cones through the sextic with* $\mathfrak{E}_{ij}$ *and* $\mathfrak{E}_{ik}$ *as vertices intersect moreover in a plane cubic which lies in the plane* $x_j - x_k = 0$.

UNIVERSITY OF ILLINOIS.

ON ELLIPTIC CYLINDER FUNCTIONS OF THE SECOND KIND.

By Sasindrachandra Dhar.

1. The canonical form of Mathieu's differential equation is given by

$$\frac{d^2y}{dz^2} + (A + 16q \cos 2z)y = 0. \tag{1}$$

For certain values of "A." two kinds of solutions of the above differential equation have been constructed. The periodic solutions of the first kind have been denoted by Professor Whittaker* in the forms:

$$\left.\begin{array}{r} ce_0(z, q),\ ce_1(z, q),\ \cdots\ ce_m(z, q),\ \cdots \\ se_1(z, q),\ \cdots\ se_m(z, q),\ \cdots \end{array}\right\}. \tag{2}$$

The solutions of the second kind corresponding to the above solutions of the first kind were first systematically studied by Mr. E. Lindsay Ince,† who gave us two methods for constructing the series of integrals. These like the solutions of the first kind are not, however, periodic. The special value of "A" for which the solution $ce_m(z, q)$ has been constructed is given by

$$A = m^2 + \frac{32q^2}{m^2 - 1} - \frac{128(5m^2 + 7)q^4}{(m^2 - 1)^3(m^2 - 4)} - \cdots, \text{ etc.} \tag{3}$$

We shall denote it, however, as

$$A = a_0 + a_1 q + a_2 q^2 + \cdots, \text{ etc.}, \tag{4}$$

where

$$a_0 = m^2, \qquad a_1 = 0, \qquad a_2 = \frac{32}{m^2 - 1}, \qquad \cdots, \qquad \text{etc.}$$

2. The existence of an infinite number of solutions of the second kind corresponding to the infinite number of solutions of the first kind can be easily demonstrated by the following well-known theorem of linear differential equation of the second order:

If $y = v$ be a particular integral of the differential equation

$$\frac{d^2y}{dz^2} + Qy = 0,$$

then the most general solution of the above equation is given by

$$y = v\left(B + C\int \frac{1}{v^2}\,dz\right).$$

* Whittaker, *Fifth International Congress of Mathematics*, 1912.

† E. Lindsay Ince, *Proc. Edin. Math. Soc.*, Vol. XXXIII, 1914–15.

Therefore the solution of the second kind corresponding to the solution $y = v$, of the first kind, is given by

$$y = Cv \int \frac{1}{v^2} dz, \tag{5}$$

where C is an arbitrary constant.

3. Mr. Ince has by using the above formula (5) calculated some of the simplest of the integrals. He has, in fact, calculated the integral corresponding to $ce_0(z, q)$. But the process is very laborious in practice and even then one cannot get as many terms as one would like.

His other method is comparatively easy, but requires for the determination of the solutions a knowledge of the forms of the integrals, which will be furnished by the formula (5). But all these integrals can be easily constructed by proceeding with a little modification, the method employed by Mathieu* and Sieger.† We proceed thus:

If, for instance, we require to construct the integral of the second kind corresponding to $ce_m(z, q)$, we will take the expression for "A" to be that given in (4) and assume

$$y = f_0(z) + qf_1(z) + q^2 f_2(z) + \cdots \text{ etc.}, \tag{6}$$

where $f_0(z), f_1(z), f_2(z), \cdots$, etc., are functions of z only.

Substituting these values of A and y in the differential equation (1) and equating the different powers of q to zero, we get the following differential equations from which to determine $f_0(z), f_1(z), f_2(z), \cdots$, etc.:

$$(7)\quad \left\{\begin{aligned} &f_0''(z) + a_0 f_0(z) = 0, \\ &f_1''(z) + a_0 f_1(z) = -\{a_1 f_0(z) + 16 \cos 2z f_0(z)\}, \\ &f_2''(z) + a_0 f_2(z) = -\{a_2 f_0(z) + a_1 f_1(z) + 16 \cos 2z f_0(z)\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &f_n''(z) + a_0 f_n(z) = -\{a_n f_0(z) + a_{n-1} f_1(z) \\ &\qquad\qquad + \cdots + a_1 f_{n-1}(z) + 16 \cos 2z f_{n-1}(z)\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right.$$

4. If, now, we solve the differential equation

$$f''(z) + a_0 f_0(z) = 0,$$

we find that there are two possible values for $f_0(z)$, viz., $\cos mz$ and $\sin mz$. If we proceed with $f_0(z) = \cos mz$, it will only enable us to obtain the solution $ce_m(z, q)$, for it is in this way that $ce_m(z, q)$ can be obtained, the constants $a_1, a_2, \cdots$, etc., being determined by the fact that $ce_m(z, q)$ is to be periodic.

* E. L. Mathieu, *Liouville's Journal* (2), XIII (1868).

† Sieger, *Annalen der Physik*, Bd. 27.

We will, therefore, proceed with $f_0(z) = \sin mz$, and solve the equations one by one.

In determining $f_1(z), f_2(z), \cdots$, etc., from the differential equations given in (7), it will be found necessary to put the expressions on the right-hand side of those equations in series of cosines or sines of multiples of z and further it will be seen that we shall be constantly required to find particular integrals of equations of the types:

$$\left.\begin{array}{ll} \text{(i) } y'' + m^2y = C\cos(m+\alpha)z, & \text{(ii) } y'' + m^2y = C\sin(m+\alpha)z, \\ \text{(iii) } y'' + m^2y = C\cos mz, & \text{(iv) } y'' + m^2y = C\sin mz, \\ \text{(v) } y'' + m^2y = Cz\cos\alpha z, & \text{(vi) } y'' + m^2y = Cz\sin\alpha z, \end{array}\right\} \quad (8)$$

whose particular integrals are given by

$$\left.\begin{array}{l} \text{(i) } y = -C\cos(m+\alpha)z/\alpha(2m+\alpha), \\ \text{(ii) } y = -C\sin(m+\alpha)z/\alpha(2m+\alpha), \\ \text{(iii) } y = Cz\sin mz/2m, \\ \text{(iv) } y = -Cz\cos mz/2m, \\ \text{(v) } y = \dfrac{C}{m^2-\alpha^2}z\cos\alpha z + \dfrac{2C\alpha}{(m^2-\alpha^2)^2}\sin\alpha z, \\ \text{(vi) } y = \dfrac{C}{m^2-\alpha^2}z\sin\alpha z - \dfrac{2C\alpha}{(m^2-\alpha^2)^2}\cos\alpha z. \end{array}\right\} \quad (9)$$

Thus, following the above processes, all the integrals of the second kind corresponding to those of the first kind as given in (2) can be obtained very easily. Following the notation suggested by Professor Whittaker, they may be denoted as

$$\left.\begin{array}{l} i\eta_0(z,q),\ i\eta_1(z,q),\ i\eta_2(z,q),\ \cdots\ i\eta_m(z,q),\ \cdots \\ \qquad\quad j\eta_1(z,q),\ j\eta_2(z,q),\ \cdots\ j\eta_m(z,q),\ \cdots. \end{array}\right\} \quad (10)$$

5. Let us illustrate the processes indicated above by working out a particular case. Suppose we wish to find the integral which corresponds to $ce_1(z,q)$. The particular value of "A" for this is given by

$$A = 1 - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \frac{88}{9}q^5 + \cdots, \text{ etc.} \quad (11)$$

Hence to find $i\eta_1(z,q)$, we shall have to solve the equations:

$$\left.\begin{array}{l} \text{(i) } f_1''(z) + f_1(z) = 8\sin z - 16\cos 2z\cdot\sin z, \\ \text{(ii) } f_2''(z) + f_2(z) = 8\sin z + 8f_1(z) - 16\cos 2z\cdot f_1(z), \\ \text{(iii) } f_3''(z) + f_3(z) = -8\sin z + 8f_1(z) + 8f_2(z) - 16\cos 2z\cdot f_2(z), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} \quad (12)$$

(*a*) Now, the solution of the equation (i) is obtained by adding up the particular integrals of

$$y''(z) + y(z) = 16\sin z; \qquad y''(z) + y(z) = -8\sin 3z,$$

which by the help of (8) and (9) is given by

$$f_1(z) = -8z \cos z + \sin 3z.$$

(*b*) To find $f_2(z)$, we need only find the particular integrals of the following equations:

$$y''(z) + y(z) = 64z \cos 3z; \qquad y''(z) + y(z) = 8 \sin 3z;$$

and

$$y''(z) + y(z) = -8 \sin 5z,$$

and add them up. Thus we get

$$f_2(z) = 8z \cos 3z + 5 \sin 3z + \frac{1}{3} \sin 5z.$$

(*c*) To find $f_3(z)$, we transform the expression on the right-hand side of (iii) (12) as

$$-48 \sin z + \frac{136}{3} \sin 3z - \frac{112}{3} \sin 5z - \frac{8}{3} \sin 7z - 16z \cos 3z - 64z \cos 5z,$$

and the form of $f_3(z)$ is determined by adding up the particular integrals of
$y''(z) + y(z) = -48 \sin z, \cdots$, etc., $\quad y''(z) + y(z) = -64z \cos 5z$,
and hence we have

$$f_3(z) = 24z \cos z + 8z \cos 3z - \frac{8}{3} z \cos 5z + \frac{1}{18} \sin 7z + \frac{8}{3} \sin 5z - \frac{35}{3} \sin 3z.$$

Proceeding thus, we can get as many terms as we like and hence on arranging, we find $i\eta_1(z, q)$ to be given by

$$-8q(1 - 3q^2 + \cdots)z \left\{ \cos z + q \cos 3z + q^2 \left(-\cos 3z + \frac{1}{3} \cos 5z \right) + \cdots \right\}$$
$$+ \sin z + q \sin 3z + q^2 \left(\frac{1}{3} \sin 5z + \sin 3z \right)$$
$$+ q^3 \left(\frac{1}{18} \sin 7z + \frac{8}{3} \sin 5z - \frac{35}{3} \sin 3z \right) + \cdots, \text{ etc.}$$

ON A NEW METHOD OF CONSTRUCTING SOLUTIONS OF THE SECOND KIND.

6. The methods given above and as also employed by Mr. E. Lindsay Ince are not suitable for studying the convergence of the series; but what is given below, while allowing us to construct the series of integrals very easily, is also suitable for the consideration of their convergency. This latter method follows lines similar to that employed by Frobenius* in solving linear differential equations and also similar to that employed by Professors Whittaker and Watson† for constructing integrals of the first kind.

* Frobenius, *Crelle's Journal*, Vol. LXXVI.

† Whittaker and Watson, "Modern Analysis," pp. 413–415.

Let us investigate the solution $i\eta_m(z, q)$ corresponding to the solution $ce_m(z, q)$ of the first kind, for which the value of "A" is given by (4).

If, now, we put in Matthieu's differential equation (1)

$$A = m^2 + 8p,$$

it will become

$$\frac{d^2y}{dz^2} + m^2y = -8(p + 2q\cos 2z)y.$$

When "p" and "q" are neglected, solutions of the equation are given by

$$y = \cos mz \quad \text{and} \quad y = \sin mz.$$

If we proceed with $y = \cos mz$, it will enable us to construct the solution $ce_m(z, q)$* and so we proceed with $y = \sin mz$. Let us denote $U_0(z) = \sin mz$. Then to obtain a closer approximation, we write $-8(p + 2q\cos 2z)U_0(z)$ as a series of sines of multiples of "z" in the form

$$-8\{q\sin(m-2)z + p\sin mz + q\sin(m+2)z\}$$

which we will denote by $V_1(z)$.

Then instead of solving the differential equation

$$\frac{d^2y}{dz^2} + m^2y = V_1(z),$$

we will solve the equation

$$\frac{d^2y}{dz^2} + m^2y = W_1(z), \tag{13}$$

where $W_1(z) = V_1(z) + (8p - a_1q)\sin mz$. Its integral, which we will denote by $U_1(z)$, is given by

$$U_1(z) = \frac{-2q\sin(m-2)z}{m-1} + \frac{2q\sin(m+2)z}{m+1} + \frac{a_1qz\cos mz}{2m}. \tag{14}$$

To obtain a still closer approximation, we will express $-8(p + 2q\cos 2z)U_1(z)$ as a series of sines of multiples of "z," which we will denote by $V_2(z)$, viz.,

$$\begin{aligned} V_2(z) = {} & \frac{16q^2\sin(m-4)z}{m-1} + \frac{16pq\sin(m-2)z}{m-1} + \frac{32q^2\sin mz}{m^2-1} \\ & - \frac{16pq\sin(m+2)z}{m+1} - \frac{16q^2\sin(m+4)z}{m+1} - \frac{8q^2a_1z\cos(m-2)z}{2m} \\ & - \frac{8pqa_1z\cos mz}{2m} - \frac{8q^2a_1z\cos(m+2)z}{2m}. \end{aligned} \tag{15}$$

Here again we solve the equation

$$\frac{d^2y}{dz^2} + m^2y = W_2(z),$$

* "Modern Analysis," p. 413.

where $W_2(z) = V_2(z) - a_2q^2 \sin mz + \lambda_2 z \cos mz$, where λ_2 has been determined in such a way that $W_2(z)$ does not involve "$z \cos mz$." *

Suppose $U_2(z)$ is the integral of the above equation. We will now proceed exactly with $U_2(z)$ as we have done with $U_1(z)$ and obtain the integral $U_3(z)$.

7. Continuing thus, we get the integrals $U_0(z)$, $U_1(z)$, $U_2(z)$, $\cdots$ $U_n(z)$, $\cdots$, etc., of the differential equations

$$\left.\begin{array}{l} \dfrac{dz^2}{d^2y} + m^2y = 0, \\ \dfrac{d^2y}{dz^2} + m^2y = W_1(z), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \dfrac{d^2y}{dz^2} + m^2y = W_n(z), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \tag{16}$$

respectively, where

$$W_n(z) = V_n(z) - a_nq^n \sin mz + \lambda_n z \cos mz; \quad (m > 2)$$
$$V_n(z) = -8(p + 2q \cos 2z)U_{n-1}(z), \quad (n \geq 1).$$

Therefore, from (16), we have

$$\left(\frac{d^2}{dz^2} + m^2\right) \cdot \sum_{n=0}^{\infty} U_n(z) = \sum_{n=1}^{\infty} W_n(z),$$

$$\text{i.e.} = \sum_{n=1}^{\infty} V_n(z) + \left(8p - \sum_{1}^{\infty} a_nq^n\right) \sin mz + \sum_{n=2}^{\infty} \lambda_n z \cos mz,$$

or

$$\left\{\frac{d^2}{dz^2} + (A + 16q \cos 2z)\right\} \cdot \sum_{0}^{\infty} U_n(z)$$
$$= \left(8p - \sum_{1}^{\infty} a_nq^n\right) \sin mz + \sum_{2}^{\infty} \lambda_n z \cos mz.$$

But we have from (4)

$$8p = \sum_{1}^{\infty} a_nq^n,$$

and it will also be found that $\sum_2^\infty \lambda_n$ vanishes for the above value of "p." Hence if the series $\sum_0^\infty U_n(z)$ be uniformly convergent, the series will be a solution of Mathieu's equation. It is, in fact, the solution of the second kind, corresponding to $ce_m(z, q)$, as has been obtained by Mr. Lindsay Ince, that is,

$$i\eta_m(z, q) = \sum_{n=0}^{\infty} U_n(z). \tag{17}$$

* It will be found that $z \cos mz$ first appears in $U_m(z)$, i.e. in $V_{m+1}(z)$.

8. We now proceed to show that $\sum_2^\infty \lambda_n$ actually vanishes by working out a few particular cases. Suppose we construct the integral which corresponds to $ce_1(z, q)$, the particular value of "A" for which $ce_1(z, q)$ was obtained being given by (11), viz.,

$$A = 1 - 8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \frac{88}{9}q^5 + \cdots, \text{ etc.}$$

If we write

$$A = 1 + 8p,$$

Mathieu's differential equation reduces to

$$\frac{d^2y}{dz^2} + y = -8(p + 2q\cos 2z)y.$$

In this case $U_0(z)$ is evidently $\sin z$.

(*a*) To get $U_1(z)$, we express $-8(p + 2q\cos 2z)U_0(z)$ in a series of sines in the form

$$-8(p - q)\sin z - 8q\sin 3z = V_1(z),$$

and solve the equation

$$\frac{d^2y}{dz^2} + y = W_1(z),$$

where $W_1(z) = V_1(z) + 8(p + q)\sin z$, since there is no term $z\cos z$ contained in $V_1(z)$.

The integral of the above equation is found to be

$$U_1(z) = -8qz\cos z + q\sin 3z.$$

(*b*) Again, we express $-8(p + 2q\cos z)U_1(z)$ in a series of sines of multiples of z, thus

$$64q(p + q)z\cos z + 64q^2z\cos 3z - 8pq\sin 3z - 8q^2\sin 5z - 8q^2\sin z$$

which we denote by $V_2(z)$.

Then we shall have to find an integral of the differential equation

$$\frac{d^2y}{dz^2} + y = W_2(z),$$

where $W_2(z) = V_2(z) - 64q(p + q)z\cos z + 8q^2\sin z$ (here $W_2(z)$ is made independent of $z\cos z$). On simplification, we get

$$W_2(z) = 64q^2z\cos 3z - 8pq\sin 3z + 8q^2\sin 5z.$$

The integral of the above equation is given by

$$U_2(z) = -8q^2z\cos 3z + q(6q + p)\sin 3z + \frac{1}{3}q^2\sin 5z.$$

(*c*) Again, since

$$-8(p+2q\cos 2z)U_2(z)$$
$$= 64q^3z\cos 5z + 64pq^2z\cos 3z + 64q^3z\cos z$$
$$-8q^2(6q+p)\sin z - \{8pq(6q+p) + \frac{8}{3}q^3\}\sin 3z$$
$$-8\{q^2(6q+p) + \frac{1}{3}pq^2\}\sin 5z - \frac{8}{3}q^3\sin 7z,$$

which we denote by $V_3(z)$, we have

$$W_3(z) = V_3(z) - 64q^3z\cos z - 8q^3\sin z,$$

so that $W_3(z)$ may not contain $z\cos z$.

Now, solving the differential equation

$$\frac{d^2y}{dz^2} + y = W_3(z),$$

we find

$$U_3(z) = -\frac{8}{3}q^3z\cos 5z - 8pq^2z\cos 3z + 4q^2(7q+p)z\cos z$$
$$+ q\left(p^2 + 12pq + \frac{1}{3}q^2\right)\sin 3z + \frac{4}{9}q^2(p+7q)\sin 5z + \frac{1}{18}q^3\sin 7z.$$

(*d*) In the same way, $V_4(z) = -8(p+2q\cos 2z)U_3(z)$. Now, if we express it in a series of sines, we find that the term which contains $z\cos z$ as a factor is

$$-32q^2(p^2+6pq+7q^2)z\cos z,$$

and the term which contains $\sin z$ is

$$-8q^2\left(p^2 + 12pq + \frac{1}{3}q^2\right)\sin z.$$

Hence we define $W_4(z)$ such that

$$W_4(z) = V_4(z) + 32q^2(p+6pq+7q^2)z\cos z + \frac{8}{3}q^4\sin z.$$

$W_4(z)$ is thus made independent of $z\cos z$. We can now find the integral $U_4(z)$.

Proceeding thus, we can find all the integrals $U_0(z)$, $U_1(z)$, $U_2(z)$, $U_3(z)$, $\cdots U_n(z)$, $\cdots$, etc. Hence if, in the series

$$U_0(z) + U_1(z) + U_2(z) + \cdots + U_n(z) + \cdots,$$

as found above, we substitute by (11)

$$8p = -8q - 8q^2 + 8q^3 - \frac{8}{3}q^4 - \cdots, \text{ etc.},$$

we get, after arranging,

$$-8q(1-3q^2+\cdots)z\{\cos z+q\cos 3z+q^2(-\cos 3z+\tfrac{1}{8}\cos 5z)+\cdots\}$$
$$+\sin z+q\sin 3z+q^2(\tfrac{1}{3}\sin 5z+5\sin 3z)$$
$$+q^3\left(\frac{1}{18}\sin 7z+\frac{8}{3}\sin 5z-\frac{35}{3}\sin 3z\right)+\cdots \tag{18}$$

correct to the third power of q.

And here it will be noticed that $\sum_2^\infty\lambda_n$ as obtained above is given by

$$-64q(p+q)-64q^3+32q^2(p^2+6pq+7q^2)+\cdots,$$

which, on substitution of the value of "p" in terms of "q," vanishes if we neglect terms containing higher powers of "q" than the fourth.

Hence the series (18) is a solution of the differential equation (1) and is the solution of the second kind denoted by $i\eta_1(z, q)$.

9. Similarly if we proceed to construct the integral corresponding to the value of "A" for $ce_2(z, q)$, viz.,

$$A=4+\frac{80}{3}q^2-\frac{6104}{27}q^4+\cdots,\ \text{etc.},$$

we find

$$U_0(z)=\sin 2z,$$
$$U_1(z)=\frac{2}{3}q\sin 4z,$$
$$U_2(z)=\frac{1}{6}q^2\sin 6z+\frac{4}{9}pq\sin 4z+8q^2z\cos 2z,$$
$$U_3(z)=\frac{1}{45}q^3\sin 8z+\frac{11}{72}pq^2\sin 6z+\frac{1}{9}q\left(\frac{8}{3}p^2-31q^2\right)\sin 4z$$
$$-16q^3z+\tfrac{8}{3}pq^2z\cos 2z+\tfrac{16}{3}q^3z\cos 4z,$$

. . . .

. . . .

and

$$\sum\lambda_n=64pq^2+\left(\frac{64}{9}p^2q^2-256q^4+\frac{128}{3}q^4\right)+\cdots,\ \text{etc.}$$
$$=8q^2\left(\frac{80}{3}q^2+\cdots\right)+\frac{1}{9}q^2\left(\frac{80}{3}q^3+\cdots\right)^2-256q^4+\frac{128}{3}q^4+\cdots$$
$$=0,\ \text{neglecting higher powers of } q \text{ than the fourth.}$$

Thus $\sum\lambda_n$ vanishes and $\sum U_n(z)$ gives the solution of the second kind corresponding to $ce_2(z, q)$. It is evidently the solution $i\eta_2(z, q)$.

10. Again, if we proceed to construct the integral of the second kind corresponding to the integral $ce_3(z, q)$, for which

$$A=9+4q^2-8q^3+\frac{13}{5}q^4+\cdots,\ \text{etc.},$$

we obtain, in this case,

$$U_0(z) = \sin 3z,$$

$$U_1(z) = \frac{1}{2}q \sin 5z - q \sin z,$$

$$U_2(z) = \frac{1}{10}q^2 \sin 7z + \frac{1}{4}pq \sin 5z + q(p-q) \sin z,$$

$$U_3(z) = \frac{1}{90}q^3 \sin 9z + \frac{7}{100}pq^3 \sin 7z + \frac{1}{8}q\left(p^2 + \frac{2}{5}q^2\right) \sin 5z + \frac{1}{3}q^2(5p-8q)z \cos 3z - q(p-q)^2 \sin z,$$

. . . .

. . . .

and

$$\sum \lambda_n = \frac{8}{3}pq^2(5p-8q) + \left(\frac{32}{5}q^5 - \frac{40}{3}pq^4 + \frac{64}{3}p^2q^3 - \frac{28}{3}p^3q^2\right) + \cdots, \text{ etc.}$$

$$= \frac{5}{24}\{4q^2 - 8q^3 + \cdots\}^2q^2 - \frac{8}{3}\{4q^2 - 8q^3 + \cdots\}q^3 + \frac{32}{5}q^5 - \frac{5}{3}\{4q^2 - 8q^3 + \cdots\}q^4 + \frac{1}{3}\{4q^2 - 8q^3 + \cdots\}^2q^3 - \frac{7}{384}\{4q^2 - 8q^3 + \cdots\}^3q^2 + \cdots, \text{ etc.}$$

$= 0$, neglecting higher powers of q than the fifth.

Hence $\sum U_n(z)$ as obtained from above is the second solution corresponding to $ce_3(z, q)$ and is denoted by $i\eta_3(z, q)$.

That $\sum \lambda_n$ vanishes, has been further verified in a few other cases.

ON THE CONVERGENCE OF THE SERIES OF INTEGRALS OF THE SECOND KIND.

11. The process of term-by-term differentiation which we have carried out in § 7 is only permissible when we have proved that the infinite series $\sum U_n(z)$ is a uniformly convergent series of analytic functions. It is, therefore, necessary for us to examine the solution $\sum U_n(z)$ more closely with a view to study its convergence.

The forms of $U_n(z)$ which are solutions of the differential equation

$$\frac{d^2y}{dz^2} + m^2y = W_n(z)$$

will be of the following types:

(i) when $n < m$,

$$U_n(z) = {}^{*}\sum_{r=1}^{n}{}' \beta_{n,r} \sin (m-2r)z + \sum_{r=1}^{m} \alpha_{n,r} \sin (m+2r)z,$$

* Σ' means that the summation ceases at the greatest value of r, which is less than or equal to $m/2$.

(ii) when $n = m$,

$$U_m(z) = {}^{*}\sum_{r=1}^{m}{}' \beta_{m,\,r} \sin (m - 2r)z + \sum_{r=1}^{m} \alpha_{m,\,r} \sin (m + 2r)z + \delta_{0,\,0} z \cos mz,$$

(iii) when $n > m$, i.e., when $n = m + \eta$ and $\eta \geq 1$,

$$U_{m+\eta}(z) = {}^{*}\sum_{r=1}^{m+\eta}{}' \beta_{m+\eta,\,r} \sin (m - 2r)z + \sum_{r=1}^{m+\eta} \alpha_{m+\eta,\,r} \sin (m + 2r)z$$

$$+ z \left\{ \sum_{r=1}^{\eta}{}' \gamma_{\eta,\,r} \cos (m - 2r)z + \sum_{r=0}^{\eta} \delta_{\eta,\,r} \cos (m + 2r)z \right\}.$$

Then since

$$\left(\frac{d^2}{dz^2} + m^2\right) \cdot U_{m+\eta+1}(z) = - 8(p + 2q \cos 2z) U_{m+\eta}(z) - \lambda_{m+\eta+1} z \cos mz$$
$$- a_n q^n \sin mz,$$

we get on equating the coefficients of $z \cos (m + 2r)z$, $z \cos (m - 2r)z$, $\sin (m + 2r)z$, and $\sin (m - 2r)z$, the following recurrence-formulæ:

(*a*) $\quad r(m + r)\delta_{\eta+1,\,r} \quad = 2\{p\delta_{\eta,\,r} + q(\delta_{\eta,\,r-1} + \delta_{\eta,\,r+1})\}, \quad (r = 1, 2, 3, \cdots)$

(*b*) $\quad r(r - m)\gamma_{\eta+1,\,r} \quad = 2\{p\gamma_{\eta,\,r} + q(\gamma_{\eta,\,r-1} + \gamma_{\eta,\,r+1})\}, \quad \left(r \leq \dfrac{m}{2}\right).$

(*c*) $\quad r(m + r)\alpha_{m+\eta+1,\,r} + \dfrac{1}{2}(m + 2r)\delta_{\eta+1,\,r}$

$$= 2\{p\alpha_{m+\eta,\,r} + q(\alpha_{m+\eta,\,r+1} + \alpha_{m+\eta,\,r-1})\} \quad \left(\begin{matrix} r = 1, 2, 3, \cdots \\ \eta \geq 1 \end{matrix}\right),$$

but when $n < m$,

$$r(m + r)\alpha_{n+1,\,r} = 2\{p\alpha_{n,\,r} + q(\alpha_{n,\,r+1} + \alpha_{n,\,r-1})\},$$

(*d*) $\quad r(r - m)\beta_{m+\eta+1,\,r} + \dfrac{1}{2}(m - 2r)\gamma_{\eta+1,\,r}$

$$= 2\{p\beta_{m+\eta,\,r} + q(\beta_{m+\eta,\,r+1} + \beta_{m+\eta,\,r-1})\}, \quad \left(r \leq \frac{m}{2}\right),$$

but when $n < m$,

$$r(r - m)\beta_{n+1,\,r} = 2\{p\beta_{n,\,r} + q(\beta_{n,\,r-1} + \beta_{n,\,r+1})\}, \quad \left(r \leq \frac{m}{2}\right),$$

with the following restrictions:

$$\alpha_{n,\,r} = \beta_{n,\,r} = \gamma_{n,\,r} = \delta_{n,\,r} = 0, \qquad \text{if} \quad r > n,$$

and also

$$\alpha_{n,\,0} = \beta_{n,\,0} = \gamma_{n,\,0} = 0, \quad \text{whatever } n \text{ is.}$$

12. If we denote

$$\sum_{\eta=r}^{\infty} \delta_{\eta,\,r} = D_r, \qquad \sum_{\eta=r}^{\infty} \gamma_{\eta,\,r} = C_r,$$

then D_r and C_r give the sum of the coefficients of $z \cos (m + 2r)z$, and $z \cos (m - 2r)z$ respectively in $\sum U_n(z)$.

Hence from (a) and (b), § 11, we get

$$r(m+r)D_r = 2\{pD_r + q(D_{r-1} + D_{r+1})\}, \tag{18}$$
$$r(r-m)C_r = 2\{pC_r + q(C_{r-1} + C_{r+1})\}. \tag{19}$$

From the forms given in (18) and (19), it is evident that the series for D_r and C_r are both convergent* and that

$$\lim_{r=\infty} D_r = 0, \quad \text{and} \quad \lim_{r=\infty} C_r = 0.$$

Similarly, denoting

$$\sum_{n=1}^{\infty} \alpha_{n,\,r} = A_r; \qquad \sum_{n=r}^{\infty} \beta_{n,\,r} = B_r,$$

which represent respectively the coefficients of sin $(m+2r)z$ and sin $(m-2r)z$, occurring in $\sum U_n(z)$, we get from (c) and (d)

$$r(m+r)A_r + \frac{1}{2}(m+2r)D_r = 2\{pA_r + q(A_{r-1} + A_{r+1})\}, \tag{20}$$

$$r(r-m)B_r + \frac{1}{2}(m-2r)C_r = 2\{pB_r + q(B_{r-1} + B_{r+1})\}. \tag{21}$$

Writing

$$\omega_r = -2q/\{r(m+r) - 2p\}, \qquad \omega_r' = -2q/\{r(r-m) - 2p\},$$
$$v_r = (m+2r)/2\{r(m+r) - 2p\}, \qquad v_r' = (m-2r)/2\{r(m+r) - 2p\},$$

we get from (20) and (21) respectively

$$v_r D_r + \omega_r A_{r-1} + A_r + \omega_r A_{r+1} = 0, \tag{22}$$
$$v_r' C_r + \omega_r' B_{r-1} + B_r + \omega_r' B_{r+1} = 0. \tag{23}$$

Eliminating $A_1, A_2, \cdots A_{r-1}, A_{r+1}, \cdots$ from (22), we get

$$A_r = \frac{(-1)^r}{\Delta_0} \square_r, \tag{24}$$

where

$$\Delta_0 = \begin{vmatrix} 1 & \omega_1 & & & & & & & \\ \omega_2 & 1 & \omega_2 & & & & & & \\ 0 & \omega_3 & 1 & \omega_3 & & & & & \\ 0 & 0 & \omega_4 & 1 & \omega_4 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

* Whittaker and Watson, "Modern Analysis," pp. 415–416.

$$\square_r = \left|\begin{array}{cccccccccccc} v_1D_1+\omega_1, & 1, & \omega_1, & & & & & & & & & \\ v_2D_2, & \omega_2, & 1, & \omega_2, & & & & & & & & \\ v_3D_3, & 0, & \omega_3, & 1, & \omega_3, & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ v_{r-1}D_{r-1}, & 0 & 0 & 0 & 0 & \omega_{r-1}, & 1, & 0 & & & & \\ v_rD_r, & 0 & 0 & 0 & 0 & 0 & \omega_r, & \omega_r, & 0 & & & \\ v_{r+1}D_{r+1}, & 0 & 0 & 0 & 0 & 0 & 0 & 1, & \omega_{r+1}, & 0 & & \\ v_{r+2}D_{r+2}, & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{r+2}, & 1, & \omega_{r+3}, & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|,$$

since $A_0 = 1$.

Expanding the determinant $\square_r$ in terms of the elements of the first column, we get

$$A_r = \frac{(-1)^r}{\triangle_0}\left\{(v_1D_1+\omega_1)M_1 + \sum_{k=2}^{\infty} v_kD_kM_k\right\}, \tag{25}$$

where the M's denote the first minors of the elements of the first column.

13. It will be easily seen that with $s \geq 1$

(i) $$M_{r+s} = (-1)^{r+s-1}\cdot\omega_r\omega_{r+1}\cdots\omega_{r+s-1}\cdot\triangle_{r+s}\left|\begin{array}{cccccc} 1 & \omega_1 & & & & \\ \omega_2 & 1 & \omega_2 & & & \\ 0 & \omega_3 & 1 & \omega_3 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \omega_{r-1} & 1 \end{array}\right|,$$

where $\triangle_{r+s}$ stands for the infinite determinant

$$\left|\begin{array}{cccccc} 1 & \omega_{r+s+1} & & & & \\ \omega_{r+s+2} & 1 & \omega_{r+s+2} & & & \\ 0 & \omega_{r+s+3} & 1 & \omega_{r+s+3} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|;$$

and also when $k < r$

(ii) $$M_k = (-1)^{k-1}\omega_{k+1}\omega_{k+2}\cdots\omega_r\cdot\triangle_r\left|\begin{array}{cccccc} 1 & \omega_1 & & & & \\ \omega_2 & 1 & \omega_2 & & & \\ 0 & \omega_3 & 1 & \omega_3 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \omega_{k-1} & 1 \end{array}\right|,$$

and further

$$\text{(iii)}\quad M_r = (-1)^{r-1}\begin{vmatrix} 1 & \omega_1 & & & & & & & \\ \omega_2 & 1 & \omega_2 & & & & & & \\ 0 & \omega_3 & 1 & \omega_3 & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \omega_{r-1} & 1 & 0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \omega_{r+1} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega_{r+2} & 1 & \omega_{r+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

The infinite determinant $\triangle_r$ is convergent, whatever 'r' is, and further that $\lim_{r=\infty} \triangle_r = 1$. Hence the M's are all finite and the series (25) is therefore convergent and converges to a finite value, if r is finite, but vanishes if r is made infinitely large.

The same may be proved for B_r by means of the relation (23).

The series $\sum U_n$ is, therefore, uniformly convergent in any bounded domain of z so that term-by-term differentiation is permissible.

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* Helge Von Koch, *Acta Mathematica*, Vol. XVI.

SYLOW SUBGROUPS IN THE GROUP OF ISOMORPHISMS OF PRIME POWER ABELIAN GROUPS.

By Harry Albert Bender.

1. Introduction.

The study of groups of isomorphisms is a comparatively recent one. C. Jordan in "Traité des substitutions" (1870), p. 56, introduced the term isomorphism in the following manner: A group G' is said to be isomorphic to another group G, if one can establish between their substitutions a correspondence such: 1° that each substitution of G corresponds to only one substitution of G', and each substitution of G' to one or more substitutions of G; 2° that the product of any two substitutions of G corresponds to the product of their respective correspondents. If G and G' are simply isomorphic they are identical as abstract groups.

The far more important concept of isomorphisms was introduced by O. Hölder and E. H. Moore in 1893–94, viz., the representation of the different automorphisms of a group by the corresponding substitutions on its operators. The totality of these substitutions constitute a group known as the group of isomorphisms. The group properties of these groups were first studied by O. Hölder and E. H. Moore, but the more important results have been contributed by G. A. Miller and W. Burnside.

In this article we shall be wholly interested in the groups of isomorphisms I of prime power abelian groups. Many of the researches on isomorphisms of prime power abelian groups have been on the representation of the group of isomorphisms as a substitution group.

The following are some of the more important abstract properties of the groups of isomorphisms that have been obtained. The number of invariant operators in the I of G, the necessary and sufficient condition that the group of isomorphisms be abelian, the necessary and sufficient condition that two operators in the I of G be commutative, and if G has for its order a power of a prime number, the method of constructing a principal series, the order of a Sylow subgroup S whose order is a power of p in the I of G, the necessary and sufficient condition that there be but one such Sylow subgroup in I, and the necessary and sufficient condition that this Sylow subgroup S be abelian. In case G is of type $(1, 1, 1, \cdots)$ the number of Sylow subgroups, whose orders are a power of p in the I of G, is known, as well as many properties of special groups.

In this article we shall establish several general theorems and relations which will aid in determining more completely the properties of the group of isomorphisms, especially with reference to Sylow subgroups whose orders are a power of p in the I of G. We shall give the total number of such Sylow subgroups in I and some of their invariant subgroups, including their commutator subgroup and the subgroup common to all of them.

The determination of all the possible orders of operators in the group of isomorphisms seems to present a very difficult problem.

In the following we have determined the order of the group of isomorphisms, and have determined a necessary and sufficient condition that the group of isomorphisms has for its order a power of a prime number. We have shown that there are but p operators of G invariant under a given Sylow subgroup S of I, and have determined the total number of p-isomorphisms in the I of G.

2. CHARACTERISTIC SUBGROUPS OF G.

If G is an abelian group of order p^m, p being a prime number, and if P represents a subgroup, whose order is a power of p, in the group of isomorphisms I of G, then G involves a series of subgroups of orders $p, p^2, p^3, p^4, \cdots, p^{m-1}$ respectively, such that each of these subgroups except the last is included in the one which follows it, and each is invariant under P. Now consider all the possible automorphisms of G in which every operator corresponds to itself multiplied either on the left or on the right by some operator in the preceding subgroup. Each of these automorphisms of G corresponds to an operator, whose order is a power of p, in the group of isomorphisms of G. W. Burnside calls the automorphisms of G, which give rise to the operators of P, p-isomorphisms.*

It is known that a necessary and sufficient condition that an operator t in the group of isomorphisms of a group G of order p^m has for its order a power of p is that t transforms every operator in the series of subgroups

$$G_0,\ G_1,\ G_2,\ G_3,\ \cdots,\ G_m \quad (G_m = G)$$

of orders $1, p, p^2, p^3, p^4, \cdots, p^m$ respectively, into itself multiplied by an operator in the preceding subgroup.

Since every non-cyclic abelian group of order p^m is the direct product of independent cyclic groups, a set of independent generators of an abelian group is commonly used with the restricted meaning that the group generated by any number of these generators has only the identity in common with the group generated by the remaining operators of the set. In what follows we shall use a set of independent generators in this restricted sense,

* W. Burnside, *Proceeding of the London Mathematical Soc.*, Ser. 2, Vol. 11, p. 225.

and suppose such a set of independent generators of G to be of orders $p^{\alpha_1}, p^{\alpha_2}, p^{\alpha_3}, \cdots, p^{\alpha_n}$ $(\alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_n > 0)$, and that the number of the independent generators of these orders is $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ respectively.

Consider the subgroups

$$H_1, H_2, H_3, H_4, \cdots, H_n, \tag{1}$$

which are respectively generated by the independent generators of the same order. Each subgroup has only the identity in common with any other subgroup, and the orders of these subgroups are $p^{\lambda_1\alpha_1}, p^{\lambda_2\alpha_2}, p^{\lambda_3\alpha_3}, \cdots, p^{\lambda_n\alpha_n}$ respectively. If s is any operator of order p^δ in H_i $(\delta \leqq \alpha_i,\ i \leqq n)$, then under the group of isomorphisms s is conjugate with the operators obtained by multiplying all the operators of order p^δ in H_i by the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of H_β whose orders do not exceed $p^{\alpha_\beta - \alpha_i + \delta}$ $(\alpha_\beta - \alpha_i + \delta > 0)$, where β takes successively the values $i + 1$, $i + 2$, $i + 3, \cdots, n$.* When $\delta = 1$, $\alpha_\beta - \alpha_i + \delta \leqq 0$, and hence the operators of order p in $\{H_1, H_2, H_3, \cdots, H_i\}$ not in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$ form a complete set of conjugate operators under the I of G $(i = 1, 2, 3, \cdots, n)$. In the special case when all the invariants are equal, then the operators of the same order form a complete set of conjugate operators under I.

A subgroup of G, which corresponds to itself in every possible automorphism of the group, is called a characteristic subgroup of G. This term was first used by Frobenius in Berliner Sitzungsberichte, 1895, p. 183.

All the possible characteristic subgroups (besides the identity) have a certain characteristic subgroup in common. This is called the fundamental characteristic subgroup of G, and is the only one in which all the operators, besides the identity, are conjugate under I. All these operators must be of order p, and must be the same power of independent generators, and hence must be the subgroup composed of all the operators of order p generated by the independent generators of highest order in a given set.

It has been shown that if s is any operator of order p^δ in H_i $(\delta \leqq \alpha_i$, $i \leqq n)$, then under I s is conjugate with the operators obtained by multiplying all the operators of order p^δ in H_i by the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of H_β whose orders do not exceed $p^{\alpha_\beta-(\alpha_i-\delta)}$ $(\alpha_\beta - \alpha_i + \delta > 0)$, where β takes successively the values $i + 1, i + 2, \cdots, n$. These operators of order p^δ are such that they generate a characteristic subgroup which is the smallest characteristic subgroup that contains all the operators of order p^δ in H_i. If we multiply all the operators of order p^σ in H_i into the group generated by all the operators whose orders do not exceed p^δ in

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. 27, p. 18.

$\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators whose orders are less than p^δ in $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$, then these operators generate a second characteristic subgroup such that in general there exists in this second characteristic subgroup many characteristic subgroups each of which contains the first as a constituent. We may determine these characteristic subgroups as follows. *A necessary and sufficient condition that a subgroup of G which contains the operators of order p^δ in H_i ($\delta \leqq \alpha_i$, $i \leqq n$), but contains no operator of higher order, be characteristic, is that it be generated by operators obtained by multiplying all the operators of order p^δ in H_i into the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators in H_β whose orders do not exceed $p^{\alpha'_\beta}$, where $\alpha'_{\beta-1} \geqq \alpha'_\beta \geqq \alpha_\beta - \alpha_{\beta-1} + \alpha'_{\beta-1}$ ($\alpha'_\beta \leqq \alpha_\beta$) as β takes successively the values $i+1, i+2, \cdots, n$ ($\alpha'_i \equiv \delta$).*

This follows at once, for if an operator s of order p^δ in H_i be multiplied by any operator which is the $p^{\alpha_{\beta-1}-\alpha'_{\beta-1}}$ power of an independent generator, then under I the operator formed by this product is conjugate with any of the operators formed by multiplying s into at least the $p^{\alpha_{\beta-1}-\alpha'_{\beta-1}}$ power of all the independent generators whose orders are less than $p^{\alpha_{\beta-1}}$ ($\beta = i+1, i+2, \cdots, n$).

The number of characteristic subgroups in G is given in the AMERICAN JOURNAL OF MATHEMATICS, Vol. 27, p. 23, and as we see does not depend on the number of independent generators of each order, but only on the different orders of the independent generators, while the orders of these characteristic subgroups are dependent on the number of independent generators of each order, as well as the orders of the independent generators.

3. SUCCESSIVE SUBGROUPS OF G INVARIANT UNDER I.

We shall represent a series of subgroups of G, whose orders are $1, p, p^2, p^3, \cdots, p^m$ respectively, such that each subgroup except the last is included in the one which follows it, and each is invariant under a given Sylow subgroup, whose order is a power of p in the I of G, by

$$G_0, G_1, G_2, G_3, \cdots, G_m \quad (G_m \equiv G). \tag{2}$$

Let S represent this Sylow subgroup, and suppose the order of S to be p^M. Such a Sylow subgroup is sometimes called the group of p-isomorphisms of G.*

Furthermore we shall suppose this series of subgroups to be selected by the method commonly used, viz., G_1 is any one of the subgroups of order p generated by an independent generator of highest order, and $G_0, G_1, G_2, \cdots, G_{\lambda_1}$ contain all the operators of order p generated respectively by the

* Cf. the first footnote.

$1, 2, 3, \cdots, \lambda_1$ independent generators of highest order, and G_{λ_1+1} is generated by G_{λ_1} and the subgroup of order p generated by an arbitrary independent generator of next largest order. Continue this arrangement so that all the operators of order p appear first, followed by all the operators of orders $p^2, p^3, p^4, \cdots, p^{\alpha_1}$ in the order named.

Such a series of subgroups is called a principal series of subgroups corresponding to the Sylow subgroup S, and it is known that S has only one principal series of subgroups.

It is evident that the successive characteristic subgroups in the principal series $G_0, G_1, G_2, \cdots, G_m$ are the subgroups composed of the identity and all the operators of order p in $\{H_1\}, \{H_1, H_2\}, \cdots, \{H_1, H_2, H_3, \cdots, H_n\}$ respectively, and in general all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_i\}$ multiplied by all the operators whose orders are less than p^δ in $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$.

Thus the following subgroups of the principal series (2) are characteristic subgroups

$$(3)\qquad \begin{gathered} G_{\lambda_1},\ G_{\lambda_1+\lambda_2},\ G_{\lambda_1+\lambda_2+\lambda_3},\ \cdots,\ G_{\lambda_1+\lambda_2+\cdots+\lambda_n},\ \cdots, \\ G_{\delta\lambda_1+\delta\lambda_2+\cdots+\delta\lambda_i+\alpha_{i+1}\lambda_{i+1}+\cdots+\alpha_n\lambda_n},\ \cdots \quad (\alpha_i \geqq \delta > \alpha_{i+1}), \end{gathered}$$

and no other subgroup of this principal series is characteristic.

It should be noted that these subgroups are not all the characteristic subgroups of G, and that in general not all the operators of one of the characteristic subgroups in (3), not in the preceding characteristic subgroup, are conjugate under the group of isomorphisms of G.

We shall consider some of the conditions in order that a series of subgroups of G whose orders are $1, p, p^2, \cdots, p^m$ respectively, and each subgroup except the last is included in the one which follows it, shall have each subgroup invariant under the I of G.

Since all the operators of the same order in H_i $(i = 1, 2, 3, \cdots, n)$ are conjugate under I, it follows that *such a series can exist only when there is but one independent generator of each order.* We shall show this condition to be sufficient. The cyclic group of order $p^{\alpha_1-\alpha_2}$ generated by the independent generator of largest order has each subgroup invariant under I. Extend this subgroup by an operator of order p generated by the independent generator of next largest order. Extend this latter subgroup by an operator which is the p^{α_1-1} power of the independent generator of largest order, and continue by an operator which is the p^{α_2-2} power of the independent generator of next largest order. If $\alpha_2 - \alpha_3 > 1$ we continue with an operator which is the p^{α_1-2} power of the independent generator of largest order. Continue this process until all the operators whose orders divide $p^{\alpha_2-\alpha_3}$ of H_2 have been used. Then continue with an operator of order p in H_3 and repeat

as above. It is evident that this process may be continued until we have all the operators of G, and that each successive subgroup satisfies the necessary and sufficient condition stated above that a subgroup be characteristic.

Furthermore in any such series each subgroup of the series must satisfy the necessary and sufficient condition, stated above, that a subgroup of G be characteristic. That the series may be chosen in many different ways is shown by the following example. Let G be of type $(m-1, 1)$, then the series of subgroups of orders $1, p, p^2, \cdots, p^m$ respectively, such that each is invariant under the I of G, may be selected in $m-2$ ways, for we may extend the cyclic subgroup of order p^δ by the independent generator of order p for $\delta = 1, 2, 3, \cdots, m-2$.

If G contains but one independent generator of largest order, then G may contain characteristic subgroups of every order less than p^m, even though some of the independent generators are of the same order. But these characteristic subgroups are such that they do not form a series of subgroups each of which is included in the one which follows. That there be but one independent generator of largest order follows from the fact that the fundamental characteristic subgroup must be of order p, since all the operators of the fundamental characteristic subgroup, besides the identity, are conjugate under I.

A simple illustration would be the group of type $(m-2, 1, 1)$ $(m \geqq 4)$. If the characteristic cyclic subgroup of order p^δ be extended by one of the independent generators of order p, the resultant group will not be characteristic. That is, the characteristic cyclic subgroup of order p^δ when extended by one of the independent generators of order p must be extended by both of the independent generators of order p, if the resultant group is to be characteristic. Since for a cyclic characteristic subgroup δ may assume the following values $1, 2, 3, \cdots, m-3$, we have characteristic subgroups of the following orders $p, p^2, p^3, \cdots, p^{m-3}$. If at least the characteristic subgroup of order p^3 and of type $(1, 1, 1)$ be extended by operators of order $p^2, p^3, \cdots, p^{m-2}$ successively, then we have characteristic subgroups of orders $p^3, p^4, p^5, \cdots, p^m$. Thus we see that this group does contain characteristic subgroups of each order but these characteristic subgroups are such that they do not form a series of subgroups each of which is included in the one which follows.

A series of subgroups $G_0, G_1, G_2, \cdots, G_m$ such that each subgroup except the last is included in the one which follows and each is invariant under a given Sylow subgroup S of the I of G may however be selected in many different ways other than being selected as the principal series. Since for a given Sylow subgroup S in the I of G there exists a series of subgroups,

such that each operator of G is transformed under S into itself multiplied by an operator in the subgroup that precedes it, then the independent generators of like orders are not conjugate under S but each is transformed into itself multiplied by at least those independent generators that precede. Thus all the independent generators of G have a definite order of arrangement for a given S and this order, as well as the order of arrangement of the respective powers of the independent generators, must be preserved in the formation of a series.

In order that any subgroup of G which includes the operators s_1', s_2', s_3', $\cdots$, s_α^1 $(\alpha \leqq \lambda_i)$ of order p^δ $(\delta \leqq \alpha_i)$ in H_i, which are generated by α distinct independent generators of order p^{α_i}, be invariant under S, it is necessary that this subgroup at least contains all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$ as well as all the operators generated by s_1', s_2', s_3', $\cdots$, s_α^1, and all the remaining operators of H_i whose orders are less than p^δ, and all the operators in H_β whose orders do not exceed $p^{\alpha_\beta'}$, where $\alpha_{\beta-1}' \geqq \alpha_\beta' \geqq \alpha_\beta - \alpha_{\beta-1} + \alpha_{\beta-1}'$ $(\alpha_\beta' \leqq \alpha_\beta)$ as β takes successively the values $i+1, i+2, i+3, \cdots, n$ $(\alpha_i' \equiv \delta)$, $(\delta > \alpha_\beta)$.

If there is but one independent generator of each order, then any series of subgroups such that each subgroup except the last is included in the one which follows and each is invariant under I, is also invariant under S and conversely. In any group the fundamental characteristic subgroup is uniquely determined for a given Sylow subgroup in the I of G. If $\alpha_1 - \alpha_2 > 1$, the fundamental characteristic subgroup may be extended by an operator of order p^2 generated by the first independent generator of largest order or by an operator of order p generated by the first independent generator of next largest order.

The number of different ways of selecting a series of subgroups such that each subgroup except the last is included in the one which follows and each is invariant under a Sylow subgroup S of the I of G, may be best illustrated by the following examples. If G is of type $(m-2, 1, 1)$, then the number of different series is $[(m-2)(m-3)]/2!$. This follows at once for if we consider the cyclic group of order p^{m-2} to have each successive subgroup of orders $p, p^2, p^3, \cdots, p^{m-2}$ generated by $s_1, s_2, s_3, \cdots, s_{m-2}$ respectively and the two independent generators of order p to be s_{m-1} and s_m, then each successive subgroup of the series may be generated respectively by these m generators. We see that s_1 is in the fundamental characteristic subgroup and hence must generate G_1, s_{m-2} must be an operator in G_m not in G_{m-1}, and the remaining $m-2$ operators may have all possible arrangement such that $s_2, s_3, \cdots, s_{m-3}$, and s_{m-1}, s_m appear in the order named. Thus in this case the number of different ways of selecting a series of subgroups each of which is invariant under S is the number of permutations of $m-2$

things taken two at a time in a fixed order, and hence this number is $[(m-2)(m-3)]/2!$. In these $[(m-2)(m-3)]/2!$ different series it is to be noted that the series generated by $s_1, s_{m-1}, s_m, s_2, s_3, \cdots, s_{m-2}$ respectively is the principal series corresponding to S. In the simple case when G is of type $(1, 1, 1, \cdots)$ the series is uniquely determined for each Sylow subgroup S in the I of G.

4. Operators of G which are Conjugate under I.

Let us now consider the characteristic subgroup generated by all the operators of order p^δ in G. If not all the operators of order p^δ are conjugate under I, then we shall consider some of the conditions in order that a subset of the operators of order p^δ will generate a second characteristic subgroup, such that all the operators in the first subgroup which are not in the second subgroup are conjugate under I. We shall also consider the condition in order that a subset of operators of this second characteristic subgroup will generate a third characteristic subgroup, such that all the operators in the second subgroup which are not in the third are conjugate under I.

From the condition given above for a subgroup of G to be characteristic it follows that *a necessary and sufficient condition that this first conjugate set of operators of order p^δ under I be composed of the operators formed by multiplying all the operators of order p^δ in H_i into the group generated by all the operators whose orders do not exceed p^δ in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$, is that p^δ be the order of the independent generators in H_i $(i < n)$*, and *a necessary and sufficient condition that this second conjugate set of operators of order p^δ under I includes all the operators of order p^δ in H_{i-1}, is that the independent generators of H_{i-1} be exactly p times as large as the independent generators of H_i.*

The first follows from the fact that by hypothesis all the operators of H_β $(\beta = i+1, i+2, \cdots, n)$ are to be included in the operators of H_β whose orders do not exceed $p^{\alpha_\beta - \alpha_i + \delta}$, hence

$$\alpha_\beta - \alpha_i + \delta = \alpha_\beta, \qquad \text{and} \qquad \delta = \alpha_i.$$

The second follows from the fact that all the operators of H_i whose orders do not exceed $p^{(\alpha_i - \alpha_{i-1} + \delta)}$ must include the operators of order $p^{\alpha_{i-1}}$ hence

$$\alpha_i - \alpha_{i-1} + \delta = \delta - 1, \qquad \text{and} \qquad \alpha_{i-1} - \alpha_i = 1 \quad (\delta \equiv \alpha_i).$$

From what precedes if $\delta = \alpha_i$, then all the operators of order p^{α_i} in G, which are formed by multiplying all the operators of order p^{α_i} in H_i into the group generated by all the operators whose orders do not exceed p^{α_i} in $\{H_1, H_2, H_3, \cdots, H_{i-1}\}$, together with the operators of $\{H_{i+1}, H_{i+2}, \cdots, H_n\}$ $(i = 1, 2, 3, \cdots, n)$, are conjugate under I.

We see from this that any operator in $G_{m-\lambda_1+1}$ not in $G_{m-\lambda_1}$ corresponds to $p^m - p^{m-\lambda_1}$ operators under the group of isomorphisms of G, and for each of these correspondences any operator in $G_{m-\lambda_1+2}$ not in $G_{m-\lambda_1+1}$ can correspond to $p^m - p^{m-\lambda_1+1}$ operators under I, etc. If G_{m_2} is the subgroup of G in the principal series $G_0, G_1, G_2, \cdots, G_m$ which contains all the operators of G whose orders do not exceed p^{α_2}, then any operator in $G_{m_2-\lambda_2+1}$ not in $G_{m_2-\lambda_2}$ corresponds to $p^{m_2} - p^{m_2-\lambda_2}$ operators under I, independent of any previous correspondence.

Since G_{m_2} contains all the operators of G whose orders do not exceed p^{α_2} it is evident that in this principal series G_{m_2} precedes G_m by $(\alpha_1 - \alpha_2)\lambda_1$, viz.,

$$m_2 = m - (\alpha_1 - \alpha_2)\lambda_1 = (\lambda_1 + \lambda_2)\alpha_2 + \lambda_3\alpha_3 + \cdots + \lambda_n\alpha_n.$$

In general G_{m_i} contains all the operators of G whose orders do not exceed p^{α_i} $(i = 1, 2, 3, \cdots, n)$ $(m_1 \equiv m)$, and in the principal series (2) G_{m_i} precedes $G_{m_{i-1}}$ by $(\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1})(\alpha_{i-1} - \alpha_i)$, viz.,

$$\begin{aligned} m_i &= m_{i-1} - (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{i-1})(\alpha_{i-1} - \alpha_i) \\ &= (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_i)\alpha_i + \lambda_{i+1}\alpha_{i+1} + \cdots + \lambda_n\alpha_n. \end{aligned} \tag{4}$$

With this notation the order of the group of isomorphisms of G is the total product of

$$(p^{m_i} - p^{m_i-\lambda_i})(p^{m_i} - p^{m_i-\lambda_i+1}) \cdots (p^{m_i} - p^{m_i-1}) \quad (i = 1, 2, \cdots, n). \tag{5}$$

It has been stated that the automorphisms in which the operators of G correspond to themselves multiplied by operators in the subgroup that precedes are p-isomorphisms. Hence we see that $p^{m_1-\lambda_1}$ of the $p^{m_1} - p^{m_1-\lambda_1}$ automorphisms are p-isomorphisms, and $p^{m_1-\lambda_1+1}$ of the $p^{m_1} - p^{m_1-\lambda_1+1}$ automorphisms are p-isomorphisms, etc., such that these p-isomorphisms leave each subgroup of the principal series (2) invariant, and thus they generate the Sylow subgroup S of order p^M of the group of isomorphisms of G. The value of M is

$$\begin{aligned} M = (m_1 - 1) + (m_1 - 2) + \cdots + (m_1 - \lambda_1) + (m_2 - 1) \\ + \cdots + (m_n - \lambda_n).* \end{aligned} \tag{6}$$

It is to be noted however that of the $p^{m_1} - p^{m_1-\lambda_1}$ automorphisms the $p^{m_1-\lambda_1}$ p-isomorphisms so chosen do not necessarily include all the p-isomorphisms in this set but only those which are included in a given Sylow subgroup S.

5. Major Co-sets Defined and their Application.

In the following discussion we shall frequently have occasion to refer to the operators of a subgroup of the principal series $G_0, G_1, G_2, \cdots, G_m$ which are not in the preceding subgroup.

* G. A. Miller, American Journal of Mathematics, Vol. 36, p. 49.

The notation $G_\alpha - G_{\alpha-1}$ is frequently used in referring to the operators of G_α which are not contained in $G_{\alpha-1}$ $(\alpha = 1, 2, \cdots, m)$. In this article we shall use the term *major co-set* when referring to these operators and shall represent it by $G_{\alpha-1}s_\alpha$, where s_α is any operator of $G_\alpha - G_{\alpha-1}$. However in some cases it will be more convenient to represent this major co-set by C_α.

If s be any operator of order $g \cdot n$ and G be any group which contains the cyclic subgroup of order g generated by s, then with G the operators of the quotient group of the cyclic group of order $g \cdot n$ with respect to the cyclic subgroup of order g will generate $n - 1$ co-sets, such that only $\varphi(n)$ of these co-sets will contain operators of order $g \cdot n$ where $\varphi(n)$ is the number of natural numbers which do not exceed n and are prime to n. Thus in general *we may define a major co-set Gs, G being any group containing the cyclic subgroup of order g generated by s, and s is any operator of order $g \cdot n$, as the totality of operators formed by multiplying all the operators of G successively by an operator in each of the $\varphi(n)$ distinct co-sets which contains operators of order $g \cdot n$ generated by s.*

Since every Sylow subgroup of order p^M in the I of G gives rise to one and only one principal series of subgroups, it follows that the total number of such Sylow subgroups in I is the total number of principal series that may be selected from G.

If we suppose the subgroup $G_{m_1-\lambda_1}$ of order $p^{m_1-\lambda_1}$ to contain all the operators of G whose orders are less than p^{α_1}, then the quotient group of G with respect to $G_{m_1-\lambda_1}$ is of order p^{λ_1} and of type $(1, 1, 1, \cdots)$, and the major co-set $G_{m_1-\lambda_1}s_{m_1-\lambda_1+1}$ may be selected from the operators of $G - G_{m_1-\lambda_1}$ in as many distinct ways as a subgroup of order p may be selected from this quotient group, or in $(p^{\lambda_1} - 1)/(p - 1)$ distinct ways. We next form the quotient group of G with respect to $G_{m_1-\lambda_1+1}$ and it is of order p^{λ_1-1} and of type $(1, 1, 1, \cdots)$, and the major co-set $G_{m_1-\lambda_1+1}s_{m_1-\lambda_1+2}$ may be selected from the remaining $p^{m_1} - p^{m_1-\lambda_1+1}$ operators in as many distinct ways as a subgroup of order p may be selected from this second quotient group, or in $(p^{\lambda_1-1} - 1)/(p - 1)$ distinct ways. If we continue this process for each successive major co-set we see that the subgroups of orders $p^{m_1-\lambda_1+1}$, $p^{m_1-\lambda_1+2}$, $\cdots$, p^{m_1-1}, p^{m} may be selected respectively in

$$\frac{p^{\lambda_1} - 1}{p - 1}, \frac{p^{\lambda_1-1} - 1}{p - 1}, \cdots, \frac{p^2 - 1}{p - 1}, 1$$

distinct ways each of which is in a distinct principal series.

Since $G_{m_i} - G_{m_i-\lambda_i}$ includes all the independent generators of G of order p^{α_i} we see that any automorphism of the operators of $G_{m_i} - G_{m_i-\lambda_i}$ $(i = 1, 2, 3, \cdots, n)$ will completely determine the automorphism of G, thus we need only to consider the distinct ways of selecting the subgroups $G_{m_i-\lambda_i+1}$,

$G_{m_l-\lambda_l+2}, \cdots, G_{m_l}$ in order to determine the total number of principal series in G.

The quotient group of G_{m_l} with respect to $G_{m_l-\lambda_l}$ is of order p^{λ_l} and of type $(1, 1, 1, \cdots)$, and hence the major co-set $G_{m_l-\lambda_l}s_{m_l-\lambda_l+1}$ may be selected from the operators of $G_{m_l} - G_{m_l-\lambda_l}$ in as many distinct ways as a subgroup of order p may be selected from this quotient group, or in $(p^{\lambda_l} - 1)/(p - 1)$ distinct ways. We may continue as above and thus determine the total number of distinct principal series in G. These results may be stated in the following theorem:

THEOREM I. *If an abelian group G of order p^m is generated by λ_1 independent generators of order p^{α_1}, in the restricted sense, λ_2 of order $p^{\alpha_2}, \cdots, \lambda_n$ of order p^{α_n} ($\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$), then the total number of Sylow subgroups of order p^M in the group of isomorphisms of G is the total product of*

$$\frac{p^{\lambda_1}-1}{p-1}\cdot\frac{p^{\lambda_2}-1}{p-1}\cdot\ldots\cdot\frac{p^{\lambda_n}-1}{p-1}$$

$$\frac{p^{\lambda_1-1}-1}{p-1}\cdot\frac{p^{\lambda_2-1}-1}{p-1}\cdot\ldots\cdot\frac{p^{\lambda_n-1}-1}{p-1}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\frac{p-1}{p-1}\cdot\frac{p-1}{p-1}\cdot\ldots\cdot\frac{p-1}{p-1}.$$

From theorem I and formulæ (5) and (6) we have the following:

THEOREM II. *If an abelian group G of order p^m is generated by λ_1 independent generators of order p^{α_1}, in the restricted sense, λ_2 of order $p^{\alpha_2}, \cdots, \lambda_n$ of order p^{α_n} ($\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$), then the order of the group of isomorphisms I of G is $(p-1)^{\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_n}$ multiplied by the product of the order of a Sylow subgroup, whose order is a power of p in I, by the total number of such Sylow subgroups in I.*

From theorems I and II we have that *a necessary and sufficient condition that the group of isomorphisms of an abelian group G of order p^m shall have for its order a power of a prime number, is that $p = 2$ and that there be but one independent generator of each order in a set of independent generators of G.*

6. OPERATORS OF G INVARIANT UNDER A SYLOW SUBGROUP OF I.

If any two operators t_1 and t_2 of the group of isomorphisms of an abelian group G transform a given operator of G into itself multiplied by s_1 and s_2 respectively, then the necessary and sufficient condition that t_1 and t_2 be commutative is that the commutator of t_1 and s_2 equals the commutator of t_2 and s_1.*

If we suppose the successive subgroups of the principal series (2) to be

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. 36, p. 47.

generated by $1, s_1, s_2, s_3, \cdots, s_m$ respectively, then we may suppose the set of independent generators of G in this discussion to be

$$(7) \qquad s_{m_n-\lambda_n+1},\ s_{m_n-\lambda_n+2},\ \cdots,\ s_{m_n},\ s_{m_{n-1}-\lambda_{n-1}+1},\ \cdots,\ s_{m_1}.$$

Furthermore, in the remaining discussion, by s_α we shall mean an operator of the set of independent generators given in (7). Hence s_α is transformed, under a Sylow subgroup S of I, into itself multiplied by every operator of the subgroup which precedes s_α. If this subgroup contains an operator s'_α of the same order as s_α, then there is no power of s_α, unless it reduces to the identity, such that all the operators of S shall be commutative with this power. For if t_1 is an operator of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha,$$

then

$$t_1^{-1}s_\alpha^n t_1 = s'^n_\alpha s^n_\alpha.$$

Since in a principal series of subgroups corresponding to S an operator s_α is preceded by the operators of the same order as s_α, which are generated by the independent generators of orders greater than that of s_α, providing s_α is not an independent generator of largest order, it follows that the only condition that not one of the commutators of s_α under S can be of the same order as s_α is that s_α be the first independent generator of largest order in the principal series corresponding to S.

That S cannot be commutative with the operators generated by s_α which are of order greater than p, follows from the fact that, under S, s_α may have for its commutator an operator whose order is one pth the order of s_α.

Thus in any abelian group G of order p^m, the subgroup G_1 in the principal series of subgroups $G_0, G_1, G_2, \cdots, G_m$ corresponding to S contains all the operators of G which are invariant under the Sylow subgroup S.

7. Invariant Operators in a Sylow Subgroup of I.

It is known that the invariant operators of I are those which transform every operator of G into the same power of itself. Of this number only those which transform every operator of G into itself multiplied by some power of p of itself are p-isomorphisms, and these p-isomorphisms are common to every Sylow subgroup, whose order is a power of p, in I. It will be assumed that this condition is sufficient and it will be shown that this necessarily includes all the operators in any Sylow subgroup S which are invariant under S, when not all the invariants of G are equal to p. In case G is of type $(1, 1, 1, \cdots)$, then a necessary and sufficient condition that an operator t_1 in a Sylow subgroup S of order p^m in I be invariant under S is that it be commutative with every operator of G_{m-1}, and the commutators

of t_1 and the operators of the major co-set $G_{m-1}s_m$ must be the operators of order p in G_1. For if

$$t_1^{-1}s_mt_1 = s_1s_m, \qquad t_2^{-1}s_mt_2 = s_\beta s_m,$$

where t_2 is any operator in the Sylow subgroup containing t_1, then we have

$$(t_1t_2)^{-1}s_mt_1t_2 = s_1s_\beta s_m, \qquad (t_2t_1)^{-1}s_mt_2t_1 = s_1s_\beta s_m.$$

It is evident that the condition is sufficient. This is equivalent to transforming every operator of G into itself multiplied by the p^{α_1-1} power of itself in case G is not of type $(1, 1, 1, \cdots)$. The following proof for the necessary condition is valid for any prime power abelian group. Suppose that

$$t_1^{-1}s_\alpha t_1 = s_\alpha' s_\alpha \quad (s_\alpha' \neq s_1,\ s_\alpha' \neq s_\alpha^{kp},\ a < m),$$
$$t_3^{-1}s_\alpha t_3 = s_\alpha, \qquad t_3^{-1}s_\alpha' t_3 = s_\beta s_\alpha' \quad (s_\beta \neq 1),$$

then

$$(t_1t_3)^{-1}s_\alpha t_1t_3 = s_\beta s_\alpha' s_\alpha, \qquad (t_3t_1)^{-1}s_\alpha t_3t_1 = s_\alpha' s_\alpha.$$

Next consider the case $s_\alpha' = s_1$. In this case t_1 either transforms all of the operators of the set of independent generators of G given by (7) into themselves multiplied by operators of order p in G_1 or is commutative with some of these generators. The possibility of t_1 transforming some of these operators into themselves multiplied by some pth power of itself is completely discussed in the next case. Let

$$t_1^{-1}s_\alpha t_1 = s_1s_\alpha, \qquad t_4^{-1}s_\alpha t_4 = s_\sigma s_\alpha,$$

then

$$(t_1t_4)^{-1}s_\alpha t_1t_4 = s_1s_\sigma s_\alpha, \qquad (t_4t_1)^{-1}s_\alpha t_4t_1 = s_1^2s_\sigma s_\alpha.$$

If t_1 is commutative with some of the operators of G, say s_α, but is not commutative with s_σ, then

$$(t_1t_4)^{-1}s_\alpha t_1t_4 = s_\sigma s_\alpha, \qquad (t_4t_1)^{-1}s_\alpha t_4t_1 = s_1s_\sigma s_\alpha.$$

The next case is that s_α' be some pth power of s_α or $s_\alpha' = s_\alpha^{kp}$. If we assume that t_1 does not transform every operator of G into itself multiplied by some pth power of itself, and suppose s_σ to be this operator, then we have

$$(t_1t_4)^{-1}s_\alpha t_1t_4 = s_\sigma^{kp}s_\alpha^{kp}s_\sigma s_\alpha, \qquad (t_4t_1)^{-1}s_\alpha t_4t_1 = t_1^{-1}s_\sigma t_1 s_\alpha^{kp}s_\alpha,$$

and thus establishing the necessary condition.

It follows from the known formula

$$(8) \qquad t^{-n}s_\alpha t^n = s_{\alpha-n}s_{\alpha-n+1}^{n} \cdots s_{\alpha-n+r}^{[n(n-1)\cdots(n-r+1)]/r!} \cdots s_{\alpha-1}^{n}s_\alpha,$$

where $t^{-1}s_\beta t = s_{\beta-1}s_\beta$ $(\beta = \alpha, \alpha-1, \alpha-2, \cdots, \alpha-n+1)$, that if $t_1^{-1}s_mt_1 = s_m^ps_m$, then t_1^p would transform s_m into itself multiplied by an operator in the major co-set containing $s_m^{p^2}$, and hence $t_1^{p\alpha_1-1} \equiv 1$. In case G

is of type $(1, 1, 1, \cdots)$,

$$t_1^{-1} s_m t_1 = s_1 s_m, \quad \text{and} \quad t_1^{-p} s_m t_1^p = s_1^p s_m,$$

and hence t_1 is of order p.

From what precedes we see that *the invariant operators of a Sylow subgroup S of order p^m in the I of G generate a cyclic subgroup of order p^{α_1-1} for $\alpha_1 > 1$, and generate a subgroup of order p for $\alpha_1 = 1$. The first being composed of the operators which transform every operator of G into itself multiplied by some* pth *power of itself, and the second is composed of the operators which are commutative with G_{m-1} and transform the operators of the major co-set $G_{m-1}s_m$ into themselves multiplied by operators of order p in G_1, and hence a Sylow subgroup whose order is a power of p in the group of isomorphisms of any abelian group G of order p^m contains but one invariant subgroup of order p.*

8. Subgroups Common to the Sylow Subgroups of I.

Since in the principal series of subgroups corresponding to any Sylow subgroup, whose order is a power of p in the group of isomorphisms of G, the subgroup $G_{m_i-\lambda_i}$ of order $p^{m_i-\lambda_i}$ will always precede all the operators of $G_{m_i} - G_{m_i-\lambda_i}$, it is evident that the automorphisms of G which make the operators of $G_{m_i} - G_{m_i-\lambda_i}$ correspond to themselves multiplied by operators in $G_{m_i-\lambda_i}$ are common to every such Sylow subgroup in I, and that no other automorphisms of these operators are common to every such Sylow subgroup $(i = 1, 2, 3, \cdots, n)$. We see this from the fact that G_{m_i} is a characteristic subgroup, and hence any p-isomorphism which makes an operator s_σ of the major co-set $G_{\sigma-1}s_\sigma$ corresponds to itself multiplied by an operator s_β in the major co-set $G_{\beta-1}s_\beta^\eta$ $(m_i - \lambda_i < \beta < \sigma \leqq m_i)$ is not in the Sylow subgroup to which the principal series generated by $1, s_1, s_2, s_3, \cdots, s_{m_i-\lambda_i}, \cdots, s_\sigma, s_\beta, \cdots, s_m$ corresponds.

Thus we have *the order of the subgroup common to every Sylow subgroup, whose order is a power of p in the group of isomorphisms of G, to be $p^{M'}$,*

$$M' = \sum_1^n (m_i - \lambda_i)\lambda_i.$$

From (6) we see that *the index of this subgroup with respect to S is $p^{M-M'}$,*

$$M - M' = \frac{1}{2}\sum_1^n (\lambda_i - 1)\lambda_i.$$

9. Invariant Subgroups in a Sylow Subgroup of I.

Consider the complete set of conjugates of an operator t_1 of S under the operators of S. Let t be any other operator of S, and suppose

$$t_1^{-1} s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t^{-1} s_\alpha t = s_\beta s_\alpha, \qquad t_1^{-1} s_\beta t_1 = s'_\beta s_\beta,$$

then

$$(tt_1t^{-1})^{-1}s_\alpha tt_1t^{-1} = tt_1^{-1}t^{-1}s_\alpha tt_1t^{-1} = ts'_\beta s'_\alpha s_\beta s_\alpha t^{-1} = ts'_\beta s'_\alpha t^{-1}s_\alpha.$$

If t be allowed to vary through all the operators of S, we see that s_β varies through all the operators of $G_{\alpha-1}$ and s'_β varies through all the commutators of $G_{\alpha-1}$ and t_1. From the expression $ts'_\beta s'_\alpha t^{-1}$ it follows that the complete set of conjugates of t_1 under S transforms s_α into itself multiplied by every operator of the major co-set containing s'_α whenever s'_α is not in the commutator subgroup of t_1 and $G_{\alpha-1}$, and in all other cases by every operator in the commutator subgroup of t_1 and $G_{\alpha-1}$.

Hence *a necessary condition that a subgroup in a Sylow subgroup S in I be invariant is that the commutators of the operators of G under the operators of this subgroup generate a group which is one of the subgroups of the principal series corresponding to S.* This is necessary for otherwise this subgroup would not contain all of its conjugates under S. However this condition is not sufficient as may be illustrated by the following example.

Suppose G to be of order p^4 and of type (1, 1, 1, 1), and suppose the principal series G_0, G_1, G_2, G_3, G_4, corresponding to S, to be generated by $1, s_1, s_2, s_3, s_4$, respectively, then the two automorphisms t_1 and t_2 of G, which are established by making

$$t_1^{-1}s_1t_1 = s_1, \quad t_1^{-1}s_2t_1 = s_2, \quad t_1^{-1}s_3t_1 = s_2s_3, \quad t_1^{-1}s_4t_1 = s_4,$$
$$t_2^{-1}s_1t_2 = s_1, \quad t_2^{-1}s_2t_2 = s_2, \quad t_2^{-1}s_3t_2 = s_2s_3, \quad t_2^{-1}s_4t_2 = s_1s_4,$$

together with the invariant subgroup of order p, will generate an abelian subgroup of order p^3. Yet this subgroup is not invariant under S even though the commutator subgroup of G under this subgroup is G_2. This subgroup is abelian because t_1 and t_2 are commutative with G_2. That this subgroup is not invariant under S is at once evident if we transform it by an operator t of S, which is commutative with s_1, s_3, and s_4 and $ts_2t^{-1} = s_1s_2$, hence

$$(tt_1t^{-1})^{-1}s_3tt_1t^{-1} = ts_2s_3t^{-1} = s_1s_2s_3$$

which is not an operator of this subgroup.

It should be noted however that the subgroup of S, composed of all the operators of S which transform the operators of G into themselves multiplied by the operators of any one of the principal series of subgroups corresponding to S, is invariant under S, but not all the operators of this subgroup are conjugate.

If a Sylow subgroup S of I be generated by operators that are commutative with every independent generator of G given by (7) *save s_j, and transform s_j into itself multiplied by s_i where j takes successively the values $m_1, m_1 - 1, \cdots, i + 1$ for each s_i and i takes successively the values* $1, 2, 3, \cdots, m - 1$, *then each sub-*

group of order p, p^2, p^3, p^4, $\cdots$, p^M respectively is an invariant subgroup of S. If s_{i+1} is not an independent generator of G, then s_j varies through only the independent generators given by (7) *which are not in G_{i+1}.*

It is evident that any one of these subgroups is composed of all the operators of S which transform the independent generators $s_{m_1}, s_{m_1-1}, \cdots, s_j$ into themselves multiplied by the operators of G_i, which transform the independent generators $s_{j-1}, \cdots, s_i$ into themselves multiplied by the operators of G_{i-1}, and which transform each of the remaining independent generators of G into itself multiplied by any operator that precedes it. If t_1 is one of these operators and t any other operator of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t^{-1}s_\alpha t = s_\beta s_\alpha, \qquad t_1^{-1}s_\beta t_1 = s'_\beta s_\beta,$$

then

$$(tt_1t^{-1})^{-1}s_\alpha tt_1t^{-1} = ts'_\alpha s'_\beta t^{-1}s_\alpha.$$

Since s_β is in $G_{\alpha-1}$ we see that all the conjugates of t_1 under S transform s_α into itself multiplied by operators of the co-set containing s'_α and hence is in this subgroup.

If the Sylow subgroup S of I be generated by operators which are commutative with every independent generator of G save s_j, and transform s_j into itself multiplied by s_i where i takes successively the values 1, 2, 3, $\cdots$, $j-1$ *for each s_j, and j takes successively the values m_1, $m_1 - 1$, $\cdots$, $m_1 - \lambda_1 + 1$, m_2, $\cdots$, $m_n - \lambda_n + 1$, then each subgroup of order p, p^2, p^3, $\cdots$, p^M respectively is an invariant subgroup of S.*

From the manner in which this Sylow subgroup is generated we see that there exists a subgroup containing t_1 which transforms any independent generator s_α into itself multiplied by all the operators of the commutator subgroup of t_1 and G_α independent of any automorphism of the remaining independent generators, and hence all the conjugates of t_1 under S are in this subgroup and thus it is an invariant subgroup.

The subgroup of order p in the two cases is common. The invariant subgroups of S of orders p^2, p^3, $\cdots$, p^{m-1} respectively are distinct in the two cases, because in the first case the operators in any one of these invariant subgroups are not commutative with the independent generator s_{m-1} of G, and in the second case all the operators of any one of these invariant subgroups are commutative with every independent generator of G except s_m. In the remaining invariant subgroups of orders p^m, p^{m+1}, $\cdots$, p^{M-1} respectively the operators in the first case do not give rise to commutators in the co-set containing s_{m-1}, while the operators in the second case for each of these invariant subgroups give rise to commutators in the co-set containing s_{m-1}. *Hence a Sylow subgroup, whose order is a power of p in the group of isomorphisms of G, contains at least $p + 1$ invariant subgroups of each order greater than p.*

Furthermore *a subgroup of S composed of all the operators of S which transform each operator of the principal series corresponding to S into itself multiplied by an operator in at least the βth major co-set which precedes the major co-set containing this operator, being commutative with the operators of G_β, is invariant under S.* For if t be any operator in this subgroup and t_1 any other operator of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t^{-1}s_\alpha t = s_{\alpha-\beta}s_\alpha, \qquad t^{-1}s'_\alpha t = s'_{\alpha-\beta}s'_\alpha,$$

then

$$(t_1 t t_1^{-1})^{-1} s_\alpha t_1 t t_1^{-1} = t_1 s'_{\alpha-\beta} s_{\alpha-\beta} t_1^{-1} s_\alpha,$$

and hence all the conjugates of t under S transform s_α into itself multiplied by operators in at least the βth major co-set which precedes the major co-set containing s_α. Thus it follows that all the conjugates under S of any operator of the subgroup composed of all the operators of S which transform every operator of G into itself multiplied by an operator in at least the βth major co-set which precedes are in this subgroup, and hence this subgroup is invariant under S.

Let t_1 and t_2 be any two operators of S such that

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha, \qquad t_2^{-1}s_\alpha t_2 = s''_\alpha s_\alpha, \qquad t_1 s''_\alpha t_1^{-1} = s''_{\alpha-1}s''_\alpha, \qquad t_2^{-1}s'_\alpha t_2 = s'_{\alpha-1}s'_\alpha,$$

then

$$(t_1 t_2 t_1^{-1} t_2^{-1})^{-1} s_\alpha t_1 t_2 t_1^{-1} t_2^{-1} = t_2 t_1 s'_{\alpha-1} s''_\alpha s'_\alpha s_\alpha t_1^{-1} t_2^{-1} = t_2 t_1 s'_{\alpha-1} t_1^{-1} s''_{\alpha-1} t_2^{-1} s_\alpha.$$

It follows from this equation that the commutator of any two operators of S transforms any operator of G into itself multiplied by an operator in at least the second major co-set which precedes it. Thus we have that the commutator subgroup of S is included in the invariant subgroup composed of all the operators which transform the operators of G into themselves multiplied by operators in at least the second major co-set which precedes them. The order of this invariant subgroup is $p^{M-m'}$,

$$M - m' = (m_1 - 2) + (m_1 - 3) + \cdots + (m_1 - \lambda_1 - 1) + (m_2 - 2) + \cdots + (m_n - \lambda_n - 1),$$

which by the aid of (6) reduces to

$$M - m' = M - (\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n), \tag{9}$$

where $m_1 - \lambda_1 - 1 = 0$ in case G is of type $(1, 1, 1, \cdots)$ and $m' = m - 1$.

If t_1 and t_2 be allowed to vary through the operators of S, we see that s'_α and s''_α will vary through all the operators of $G_{\alpha-1}$, and hence $s'_{\alpha-1}$ and $s''_{\alpha-1}$ will vary through all the possible commutators of $G_{\alpha-1}$ under S. If the major co-set $G_{\alpha-2}s_{\alpha-1}$ includes an independent generator, then the commutators of $G_{\alpha-1}$ under S will include all the operators of $G_{\alpha-2}$ and $s'_{\alpha-1}$ and $s''_{\alpha-1}$ will vary through $p^{\alpha-2}$ different operators. In case the major co-set $G_{\alpha-2}s_{\alpha-1}$ does not include an independent generator of G, then neither $s'_{\alpha-1}$ nor $s''_{\alpha-1}$ can vary through $p^{\alpha-2}$ operators.

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Suppose s_α to be the first independent generator of order p^{α_i}, then the major co-set $G_{\alpha-2}s_{\alpha-1}$ will not contain an independent generator for $\alpha_i \neq \alpha_1$. An operator in this major co-set is conjugate under S with only the operators obtained by multiplying this operator by the group generated by all the operators whose orders do not exceed p^{α_i} in $H_1, H_2, H_3, \cdots, H_{i-2}$, together with all the operators whose orders do not exceed p^{α_i} in H_{i-1}, except those operators of H_{i-1} which are in the major co-set $G_{\alpha-2}s_{\alpha-1}$, and together with all the operators of H_β whose orders do not exceed $p^{\alpha_\beta-\alpha_{i-1}+\alpha_i}$ for $\beta = i, i+1, i+2, \cdots, n$. We see that the independent generators of orders less than p^{α_i} cannot be commutators under S for the operators in the major co-set $G_{\alpha-2}s_{\alpha-1}$ $(\alpha_i \neq \alpha_1)$. Furthermore the pth power of the independent generators of order p^{α_i} cannot be commutators under S for the operators in the major co-set $G_{\alpha-2}s_{\alpha-1}$ if $\alpha_{i-1} - \alpha_i > 1$. If s'_α or s''_α is an independent generator of order less than p^{α_i}, then $s'_{\alpha-1}$ or $s''_{\alpha-1}$ can vary through every operator in the subgroup preceding s'_α or s''_α. In this case the pth power of the independent generators of order p^{α_i} cannot be commutators under S unless $\alpha_i - \alpha_{i+1} = 1$.

The order of the commutator subgroup of S is p^N, where

$$N = (m_1 - 2) + (m_1 - 3) + \cdots + (m_1 - \lambda_1) + (m_1 - \lambda_1 - 2) + (m_2 - 2) + \cdots + (m_2 - \lambda_2) + (m_2 - \lambda_2 - 3) + \cdots + (m_n - \lambda_n - 2),$$

where

$$\alpha_i - \alpha_{i+1} > 1 \qquad \text{for} \qquad i = 1, 2, 3, \cdots, n \quad (\alpha_{n+1} = 0).$$

If $\alpha_i - \alpha_{i+1} = 1$ for k different values of i, then k must be added to this value of N, whence from (6) N may be expressed as follows:

$$(10) \qquad N = M - [\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n + 2(n-1) - k].$$

The pth power of any operator t_1 of S is in the commutator subgroup. This follows directly from (8), for if

$$t_1^{-1}s_\alpha t_1 = s'_\alpha s_\alpha,$$

then

$$t_1^{-p}s_\alpha t_1^p = ss'^p_\alpha s_\alpha,$$

and the commutator ss'^p_α is evidently in the commutator subgroup, the operator s being the product of all the multiplying operators which precedes s'_α. Thus the quotient group of S with respect to the commutator subgroup is of type $(1, 1, 1, \cdots)$.

10. Total Number of p-Isomorphisms in the Group of Isomorphisms.

Let us first consider the total number of p-isomorphisms in the group of isomorphisms of the abelian group G of order p^m and of type $(1, 1, 1, \cdots)$.

We shall first consider the p-isomorphisms in I which are commutative with every operator of a given subgroup G_{m-1} of order p^{m-1}. Under these automorphisms of G an operator in the major co-set $G_{m-1}s_m$ may be made to correspond to itself multiplied by any operator in G_{m-1}, except the identity. This number is evidently

$$p^{m-1} - 1,$$

and the total number of p-isomorphisms which are commutative with p^{m-1} operators is the product of the number of p-isomorphisms commutative with a given subgroup of order p^{m-1} and the total number of subgroups of order p^{m-1} in G. Thus the total number of p-isomorphisms in I which are commutative with p^{m-1} operators is given by the following expression:

$$(11) \qquad (p^{m-1} - 1)\frac{(p^m - 1)}{(p - 1)}.$$

We now consider the p-isomorphisms commutative with a given subgroup G_{m-2} of order p^{m-2}, generated by the independent generators s_1, s_2, s_3, $\cdots$, s_{m-2}, and commutative with no other operators.

These p-isomorphisms may be divided into two classes. In the first class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-2}s_{m-1}$ give rise to commutators with which the p-isomorphisms are commutative. In the second class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-2}s_{m-1}$ give rise to commutators with which the p-isomorphisms are not commutative.

The quotient group H_2 of G with respect to G_{m-2} is of order p^2, and we may suppose it to be generated by s_{m-1}, and s_m and G to be the direct product of G_{m-2} and H_2. We see that an automorphism of any operator in a major co-set, with respect to the subgroup composed of all the commutative operators, completely establishes the automorphism of every operator of this major co-set, and hence any automorphism of the operators of H_2 will completely determine the automorphism of the group G.

For the p-isomorphisms in the first class an operator of the major co-set $G_{m-2}s_{m-1}$ can be made to correspond to itself multiplied by any operator of G_{m-2}, except the identity, or can be made to correspond to $(p^{m-2} - 1)$ operators, and for each of these automorphisms an operator in the major co-set $G_{m-1}s_m$ can be made to correspond to itself multiplied by any operator of G_{m-1}, except the operators of the subgroup of order p generated by the commutators under these p-isomorphisms of the operators of the major co-set $G_{m-2}s_{m-1}$, viz., it can have $p^{m-1} - p$ distinct automorphisms. Thus the total number of p-isomorphisms in this first class is

$$(p^{m-2} - 1)(p^{m-1} - p).$$

If the commutators of the operators of the major co-set $G_{m-2}s_{m-1}$ are not commutative with these p-isomorphisms, then they must be operators of the major co-set $G_{m-1}s_m$, and for each automorphism of an operator in the major co-set $G_{m-2}s_{m-1}$, which gives rise to one of the $p^{m-1} - p^{m-2}$ commutators in the major co-set $G_{m-2}s_m$, an operator in the major co-set $G_{m-2}s_m$ can be made to correspond to itself multiplied by any one of the $p^{m-2} - 1$ operators of G_{m-2}. Since in the quotient group H_2 we have $p + 1$ subgroups of order p, and p of these subgroups do not contain the operator s_{m-1}, we see that there are p distinct major co-sets with respect to G_{m-2} exclusive of $G_{m-2}s_{m-1}$. Hence the major co-set $G_{m-2}s_m$ may be selected from the operators of the major co-set $G_{m-1}s_m$ in p distinct ways, and the total number of p-isomorphisms in this second class is

$$(p^{m-2} - 1)(p^{m-1} - p^{m-2})p.$$

The total number of p-isomorphisms commutative with a given subgroup G_{m-2} of order p^{m-2} and commutative with no other operators of G is

$$(p^{m-2} - 1)(p^{m-1} - p) + (p^{m-2} - 1)(p^{m-1} - p^{m-2})p = (p^{m-1} - 1)(p^{m-1} - p).$$

Hence the total number of p-isomorphisms which are commutative with exactly p^{m-2} operators is

$$(12) \qquad (p^{m-1} - 1)(p^{m-1} - p)\frac{(p^m - 1)(p^{m-1} - 1)}{(p^2 - 1)(p - 1)},$$

since $\dfrac{(p^m - 1)(p^{m-1} - 1)}{(p^2 - 1)(p - 1)}$ is the number of distinct subgroups of order p^{m-2} in G, remembering that the number of subgroups of index p^α is the same as the number of subgroups of order p^α.

The number of p-isomorphisms in a given Sylow subgroup, whose order is a power of p in the group of isomorphisms of an abelian group of order p^m and of type $(1, 1, 1, \cdots)$, commutative with a given subgroup of order p^β $(\beta < m)$ and commutative with no other operators of G is given by the following expression:

$$(13) \qquad (p^\beta - 1)(p^{\beta+1} - p)(p^{\beta+2} - p^2) \cdots (p^{m-1} - p^{m-\beta-1}).$$

This is evident from the fact that any operator in the major co-set $G_{\alpha-1}s_\alpha$ $(\beta < \alpha \leqq m)$ may have for its commutator any operator in $G_{\alpha-1}$ which is not in the subgroup of order $p^{\alpha-\beta-1}$ generated by the commutators, under these p-isomorphisms, of the operators in $G_{\alpha-1} - G_\beta$. That is, for any p-isomorphisms of the operators in $G_{\alpha-1} - G_\beta$, an operator in the major co-set $G_{\alpha-1}s_\alpha$ can be made to correspond to itself multiplied by any one of the $p^{\alpha-1} - p^{\alpha-\beta-1}$ operators in $G_{\alpha-1}$ which have not been used as commutators $(\alpha = \beta + 1, \beta + 2, \cdots, m)$.

We shall now consider the p-isomorphisms commutative with a given subgroup G_{m-3}, generated by the independent generators $s_1, s_2, s_3, \cdots, s_{m-3}$, and commutative with no other operators of G. These p-isomorphisms may be divided into three classes. In the first class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-3}s_{m-2}$ give rise to commutators with which the p-isomorphisms are commutative. This number can be shown to be

$$\text{(i)} \quad (p^{m-3}-1)(p^{m-2}-p)(p^{m-1}-p^2) + (p^{m-3}-1)(p^{m-2}-p)(p^{m-1}-p^{m-2})p.$$

In the second class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-3}s_{m-2}$ give rise to commutators which themselves give rise to commutators with which the p-isomorphisms are commutative, and this number is

$$\text{(ii)} \qquad (p^{m-3}-1)(p^{m-2}-p^{m-3})(p^{m-1}-p^2)\frac{(p^3-p)}{(p-1)}.$$

In the third class we shall place all the p-isomorphisms under which the operators of the major co-set $G_{m-3}s_{m-2}$ give rise to commutators which themselves give rise to commutators with which the p-isomorphisms are not commutative, this number is

$$\text{(iii)} \quad (p^{m-3}-1)(p^{m-2}-p^{m-3})(p^{m-1}-p^{m-2})\frac{(p^3-p)(p^3-p^2)\,(p^2-1)}{(p^2-1)(p^2-p)\,\,(p-1)}.$$

It is evident that every p-isomorphism which is commutative with G_{m-3} and commutative with no other operator of G falls in one of these three classes. Hence the total number of p-isomorphisms in the group of isomorphisms of G which are commutative with a given subgroup G_{m-3} is the sum of (i), (ii), and (iii), which when reduced gives

$$(p^{m-1}-1)(p^{m-1}-p)(p^{m-1}-p^2).$$

Since the total number of subgroups of order p^{m-3} in G is

$$\frac{(p^m-1)(p^{m-1}-1)(p^{m-2}-1)}{(p-1)(p^2-1)(p^3-1)},$$

we have that the total number of p-isomorphisms in the group of isomorphisms of G which are commutative with exactly p^{m-3} operators is given by the following:

$$\text{(14)} \quad (p^{m-1}-1)(p^{m-1}-p)(p^{m-1}-p^2)\frac{(p^m-1)(p^{m-1}-1)(p^{m-2}-1)}{(p-1)(p^2-1)(p^3-1)}.$$

We have the total number of p-isomorphisms in I commutative with exactly p^{m-1} operators, exactly p^{m-2} operators, and exactly p^{m-3} operators.

The preceding method for determining the total number of p-isomorphisms in I becomes very complicated if we continue, as above, step by step. We shall show by mathematical induction that the law of (11), (12), and (14) holds for $m+1$ if it holds for m.

We shall assume that the number of p-isomorphisms in I commutative with exactly p^{m-r} operators is equal to

$$(15)\qquad (p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})\frac{(p^m-1)(p^{m-1}-1)\cdots(p^{m-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots(p-1)},$$

and we shall show that in case G is of order p^{m+1} the number of p-isomorphisms commutative with exactly p^{m-r} operators is equal to

$$(16)\qquad (p^m-1)(p^m-p)\cdots(p^m-p^r)\frac{(p^{m+1}-1)(p^m-1)\cdots(p^{m-r+1}-1)}{(p^{r+1}-1)(p^r-1)\cdots(p-1)}.$$

Formula (15) gives the total number of p-isomorphisms commutative with a subgroup of index p^r while (16) gives the total number of p-isomorphisms commutative with a subgroup of index p^{r+1}. It is evident that (16) is obtained from (15) by replacing m by $m+1$ when r is such as to give the total number of p-isomorphisms commutative with a subgroup of index p^{r+1}, viz., for r equal to $r+1$.

From (11) we have the total number of p-isomorphisms commutative with a subgroup of index p for any value of m. Likewise (12) gives the total number of p-isomorphisms commutative with a subgroup of index p^2 for any value of m. However we shall establish the latter for the case when the group is of order p^{m+1} from (15), assuming r to be equal to one.

We shall suppose the group of order p^{m+1} to be G_{m+1} and suppose it to be generated by G and s_{m+1}. We have the number of p-isomorphisms, commutative with a given subgroup G_{m-1} in the group G, to be $(p^{m-1}-1)$. For each p-isomorphism of this group G, an operator in the major co-set Gs_{m+1} can be made to correspond to itself multiplied by any operator in G which is not in the subgroup of order p generated by the commutators of G under these automorphisms. Thus the $(p^{m-1}-1)(p^m-p)$ p-isomorphisms include all the p-isomorphisms in the group of isomorphisms of the group G_{m+1} which are commutative with a given subgroup G_{m-1} of order p^{m-1} and such that the operators in the major co-set Gs_{m+1} are not used as commutators. For the remaining p-isomorphisms, which are such that the operators of the major co-set $G_{m-1}s_m$ have for their commutators operators of the major co-set Gs_{m+1}, each set of $p^m - p^{m-1}$ commutators may be selected from the operators of the major co-set Gs_{m+1} in $\dfrac{(p^2-p)}{(p-1)}$ distinct

ways. This follows from the fact that the number of distinct sets of commutators the operators of the major co-set $G_{m-1}s_m$ can have is the number of subgroups of order p in the quotient group of G_{m+1} with respect to G_{m-1} diminished by the subgroup of order p corresponding to G. An operator in each set of commutators selected from Gs_{m+1} can be made to correspond to $p^{m-1}-1$ operators, thus the total number of these p-isomorphisms is

$$(p^{m-1}-1)(p^m-p^{m-1})\frac{p^2-p}{p-1}.$$

Thus the total number of p-isomorphisms in the group of isomorphisms of G_{m+1} commutative with a given subgroup of order p^{m-1} and commutative with no other operators is

$$(p^{m-1}-1)\left[p^m-p+(p^m-p^{m-1})\frac{p^2-p}{p-1}\right]=(p^m-1)(p^m-p),$$

as was to be proved.

We shall now consider the number of p-isomorphisms commutative with a given subgroup G_{m-r} of index p^{r+1} in the group G_{m+1} and commutative with no other operators of G_{m+1} $(r=1, 2, 3, \cdots, m-1)$. From (15) we have the number of p-isomorphisms, commutative with a given subgroup G_{m-r} of order p^{m-r} in a group G of order p^m, to be

$$(p^{m-1}-1)(p^{m-1}-p)(p^{m-1}-p^2)\cdots(p^{m-1}-p^{r-1}).$$

For each of these p-isomorphisms of the group G, an operator in the major co-set Gs_{m+1} can have for its commutator any operator in G which is not in the subgroup of order p^r generated by the commutators of G under these p-isomorphisms. These automorphisms evidently include all the p-isomorphisms of the group of isomorphisms of the group G_{m+1} which are commutative with a given subgroup G_{m-r} and such that the operators in the major co-set Gs_{m+1} are not used as commutators.

In all the remaining p-isomorphisms of G_{m+1} commutative with G_{m-r}, it is necessary that some operators of the major co-set Gs_{m+1} be used as commutators. Since any two subgroups of index p have a subgroup of index p^2 in common, it follows that every subgroup of order p^m containing G_{m-r}, except G, contains exactly p^m-p^{m-1} operators of the major co-set Gs_{m+1}. The number of distinct subgroups of order p^m, different from G, that can be selected from this group of order p^{m+1}, such that each of these subgroups of order p^m contains the given subgroup G_{m-r}, is the number of subgroups of order p^r in the quotient group of G_{m+1} with respect to G_{m-r} diminished by the subgroup of order p^r corresponding to G. The quotient group is of order p^{r+1}, and the number of subgroups of order p^r is $\dfrac{p^{r+1}-1}{p-1}$.

Therefore the number of subgroups of order p^m, except G, in G_{m+1} which contains G_{m-r} is $\dfrac{p^{r+1}-p}{p-1}$.

Let us designate the operators of $G_{m-r+1} - G_{m-r}$ by C_{m-r+1}, the operators of $G_{m-r+2} - G_{m-r+1}$ by $C_{m-r+2}, \cdots$, the operators of $G_m - G_{m-1}$ by C_m.

If in any of the $(p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})$ possible p-isomorphisms, commutative with G_{m-r}, of any of the subgroups of order p^m except G, we have the commutators of the operators of C_m to be in C_{m-1}, those of C_{m-1} to be in $C_{m-2}, \cdots$, those of $C_{m-\alpha+1}$ to be in $C_{m-\alpha}$ $(\alpha < r)$, and the subgroup $G_{m-\alpha}$ to be invariant, and that $C_m, C_{m-1}, C_{m-2}, \cdots, C_{m-\alpha}$ include all the operators of the major co-set Gs_{m+1} which are in the subgroup of order p^m and that each major co-set $C_m, C_{m-1}, \cdots, C_{m-\alpha}$ contains some operators from the major co-set Gs_{m+1}, then for each such p-isomorphism of the group of order p^m we can have an operator not in this subgroup of order p^m to have for its commutator any operator of C_m. Under some of these p-isomorphisms of the group of order p^m, the operators of this subgroup do not have for their commutators any operator of the major co-set Gs_{m+1}. In such cases C_m is such as to include all the operators of the major co-set Gs_{m+1} which are in this subgroup of order p^m, and hence $\alpha = 0$. That is, if for each p-isomorphism of any subgroup of order p^m, except G, which is commutative with a given subgroup G_{m-r} of order p^{m-r} and commutative with no other operators, we allow an operator not in this subgroup of order p^m to have for its commutator any one of the $p^m - p^{m-1}$ operators in C_m, as defined above, then it can be shown that this gives all the p-isomorphisms of the group G_{m+1} of order p^{m+1} which are commutative with G_{m-r} and such that $p^m - p^{m-1}$ operators of the major co-set Gs_{m+1} are used as commutators. Therefore the total number of p-isomorphisms in the group of isomorphisms of the group G_{m+1} of order p^{m+1} and of type $(1, 1, 1, \cdots)$ which are commutative with a given subgroup of order p^{m-r} and commutative with no other operators is given by the following:

$$(p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})\left[p^m - p^r + (p^m - p^{m-1})\frac{p^{r+1}-p}{p-1}\right],$$

which when reduced gives

$$(p^m-1)(p^m-p)\cdots(p^m-p^r). \tag{17}$$

Suppose that a p-isomorphism t of the group of isomorphisms of the group G_{m+1} which is commutative with G_{m-r} and commutative with no other operators is such that

$$t^{-1}s_\beta t = s_{\beta-1}s_\beta \qquad \text{for} \qquad \beta = \beta,\ \beta-1,\ \beta-2,\ \cdots,\ \text{and}\ \beta-\alpha,$$

and leaves $G_{m-\alpha}$ invariant. There is no loss in generality in assuming the invariant subgroup $G_{m-\alpha}$ of order $p^{m-\alpha}$ to be first. If we extend this subgroup $G_{m-\alpha}$ by $s_{\beta-\alpha}, s_{\beta-\alpha+1}, \cdots, s_\beta$ respectively, then it is evident that this p-isomorphism t, which is any p-isomorphism of the group G_{m+1} which is commutative with G_{m-r} and commutative with no other operators, has been included in (17). It is also evident that if the p-isomorphism t arises from a different arrangement of the group G_{m+1} it has not been included in the total number of p-isomorphisms given by (17), viz., all the p-isomorphisms given by (17) are distinct. Thus the truth of (17) has been established as was to be proved.

Since the formula given by (17) holds for $r = 1, 2, 3, \cdots, m-1$ and with the number of subgroups of order p^{m-r} in G_{m+1} we have completely established by induction that if (15) is true for $m = m$ then it is also true for $m = m + 1$. As we have previously established the truth of the formula for $m = 2$, 3, and 4, we have the following theorems:

THEOREM III. *The total number of p-isomorphisms, commutative with a given subgroup of order p^{m-r} and commutative with no other operators of G, in the group of isomorphisms of an abelian group G of order p^m and of type* $(1, 1, 1, \cdots)$ *is*

$$(p^{m-1}-1)(p^{m-1}-p)(p^{m-1}-p^2)\cdots(p^{m-1}-p^{r-1}).$$

THEOREM IV. *The total number of p-isomorphisms in the group of isomorphisms of an abelian group G of order p^m and of type* $(1, 1, 1, \cdots)$ *is*

$$\sum_{r=1}^{m-1}(p^{m-1}-1)(p^{m-1}-p)\cdots(p^{m-1}-p^{r-1})\frac{(p^m-1)\cdots(p^{m-r+1}-1)}{(p^r-1)\cdots(p-1)}.$$

We shall now determine the total number of operators, whose orders are a power of p, in the group of isomorphisms of any abelian group G of order p^m.

We shall first consider the p-isomorphisms which are commutative with a subgroup of order p^{m-1}. Under these automorphisms of G an operator in the major co-set $G_{m-1}s_m$ can be made to correspond to itself multiplied by any operator in G_{m-1}, except the identity. This number is evidently

$$p^{m-1}-1,$$

and the total number of p-isomorphisms which are commutative with p^{m-1} operators is the product of the number of p-isomorphisms commutative with a given subgroup of order p^{m-1} and the total number of such subgroups in G. Each of these subgroups must include the subgroup $G_{m-\lambda_1}$ and hence the total number of such subgroups is the number of subgroups of index p in the quotient group of G with respect to $G_{m-\lambda_1}$. This quotient group is of order p^{λ_1} and of type $(1, 1, 1, \cdots)$. Thus we have that the total number of

p-isomorphisms in the I of G commutative with exactly p^{m-1} operators is given by the following:

$$(p^{m-1}-1)\frac{(p^{\lambda_1}-1)}{(p-1)}\,.$$

Since the automorphism of the independent generators of G completely establishes the automorphism of the group, and from the fact that all the operators of $G_{m_i} - G_{m_i-\lambda_i}$ $(i = 1, 2, 3, \cdots, n)$ are conjugate under I, we have by theorem I the total number of p-isomorphisms commutative with a given subgroup. Combining this with the number of such subgroups we have the total number of p-isomorphisms arising from the operators of $G_{m_i} - G_{m_i-\lambda_i}$, being commutative with the remaining independent generators, to be

$$\sum_{r=1}^{\lambda_i}(p^{m_i-1}-1)(p^{m_i-1}-p)\cdots(p^{m_i-1}-p^{r-1})\frac{(p^{\lambda_i}-1)\cdots(p^{\lambda_i-r+1}-1)}{(p^r-1)\cdots(p-1)}\,. \tag{18}$$

If we include the identity in this summation, then the total number of p-isomorphisms, including the identity, in the group of isomorphisms of G will be the product of these summations for $(i = 1, 2, \cdots, n)$.

Let us consider the summation

$$\sum_{r=1}^{b}(p^a-1)(p^a-p)\cdots(p^a-p^{r-1})\frac{(p^b-1)\cdots(p^{b-r+1}-1)}{(p^r-1)\cdots(p-1)} \quad (a \geqq b). \tag{19}$$

For small values of b this summation reduces to $p^{ab}-1$. We shall show by induction that this is true for any value of b less than or equal to a. Suppose

$$\sum_{r=1}^{b}(p^a-1)(p^a-p)\cdots(p^a-p^{r-1})\frac{(p^b-1)\cdots(p^{b-r+1}-1)}{(p^r-1)\cdots(p-1)} = p^{ab}-1, \tag{20}$$

and suppose that for $b = b+1$ we have

$$\sum_{r=1}^{b+1}(p^a-1)(p^a-p)\cdots(p^a-p^{r-1})\frac{(p^{b+1}-1)\cdots(p^{b-r+2}-1)}{(p^r-1)\cdots(p-1)} = p^{a(b+1)}-1. \tag{21}$$

If from the $(r+1)$th term of (21) we subtract the product of p^a and the

rth term of (20), we have

$$(p^a-1)\frac{(p^{b+1}-1)}{(p-1)}+\sum_{r=1}^{b}(p^a-1)$$

$$\cdots(p^a-p^{r-1})\frac{(p^b-1)\cdots(p^{b-r+1}-1)}{(p^r-1)\cdots(p-1)} \tag{22}$$

$$\times\left[(p^a-p^r)\frac{(p^{b+1}-1)}{(p^{r+1}-1)}-p^a\right]=p^a-1.$$

For brevity we shall designate the rth term of (20) by U_r ($r=1, 2, 3, \cdots, b$). With this we have the last term of (22) to be

$$-p^bU_b,$$

and the sum of the last two terms of (22) is

$$U_{b-1}\left[(p^a-p^{b-1})\frac{(p^{b+1}-1)}{(p^b-1)}-p^a-\frac{p^b(p^a-p^{b-1})(p-1)}{(p^b-1)}\right]$$
$$=-p^{b-1}U_{b-1}.$$

If to this we add the third from the last term, and etc., we shall assume the sum of the last $b-r+1$ terms to be equal to

$$-p^rU_r,$$

and if to this sum we add the $b-r+2$th term from the last, we have

$$U_{r-1}\left[(p^a-p^{r-1})\frac{(p^{b+1}-1)}{(p^r-1)}-p^a-\frac{p^r(p^a-p^{r-1})(p^{b-r+1}-1)}{(p^r-1)}\right]$$
$$=-p^{r-1}U_{r-1}.$$

Thus by induction we have the summation in (22) to be

$$-pU_1=-\frac{p(p^a-1)(p^b-1)}{(p-1)},$$

and hence

$$(p^a-1)\frac{(p^{b+1}-1)}{(p-1)}-\frac{p(p^a-1)(p^b-1)}{(p-1)}=p^a-1,$$

as was to be proved.

From (18) and (20) we have

$$1+\sum_{r=1}^{\lambda_1}(p^{m_1-1}-1)(p^{m_1-1}-p)$$
$$\cdots(p^{m_1-1}-p^{r-1})\frac{(p^{\lambda_1}-1)\cdot(p^{\lambda_1-r+1}-1)}{(p^r-1)\cdots(p-1)}=p^{(m_1-1)\lambda_1},$$

and hence the following theorem:

THEOREM V. *If an abelian group G of order p^m is generated by λ_1 independent generators of order p^{α_1}, in the restricted sense, λ_2 of order p^{α_2}, $\cdots$, λ_n of order p^{α_n} $(\alpha_1>\alpha_2>\cdots\geqq\alpha_n>0)$, then the total number of p-iso-*

morphisms in the group of isomorphisms of G, including the identity, is

$$p^{\sum_1^n (m_i - 1)\lambda_i}.$$

In case *the group G of order p^m is of type* (1, 1, 1, $\cdots$), *we have the unique result that the total number of p-isomorphisms, including the identity, in the group of isomorphisms of G is the square of the order of a Sylow subgroup whose order is a power of p in the group of isomorphisms.*

From (6) and theorem V we have a new proof of the theorem* that a necessary and sufficient condition that the group of isomorphisms of an abelian group G of order p^m contains but one Sylow subgroup S of order p^M is that G does not contain two equal invariants.

The statement in the *Transactions of the American Mathematical Society*, Vol. 12, p. 397, that a necessary and sufficient condition that the group of isomorphisms of a group of order p^m involves only one Sylow subgroup of order p^M is that this group of order p^m involves a characteristic subgroup of order p^γ, for every value of γ from 1 to $m - 1$ is incorrect. The condition is necessary but not sufficient as may be seen in the case G is of type ($m - 2$, 1, 1) discussed in 3.

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. 36, p. 47.

THE ISODYADIC QUINTIC.

By J. C. Glashan.

Composition.—If the roots of the quintic

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0 \tag{1}$$

are expressible by radicals they will be of the form

$$\omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4, \tag{2}$$

in which

$$(\omega^5 - 1)/(\omega - 1) = 0, \qquad n = 1, 2, 3, 4, 5.$$

Also $(y_1 y_4 - y_3 y_2)^2$ must be rational and satisfy the relation

$$\begin{aligned}\{3125(y_1y_4 - y_3y_2)^6 - 2500A(y_1y_4 - y_3y_2)^4 \\ + 400B(y_1y_4 - y_3y_2)^2 - 64C\}^2 \\ - 1024(J^2 - 128K)(y_1y_4 - y_3y_2)^2 = 0,\end{aligned} \tag{3}$$

in which

$$\begin{aligned} A &= 3c^2 + e, \\ B &= 15c^4 - 2c^2e + 8cd^2 - 2df + 3e^2, \\ C &= (5c^3 - 3ce + 4d^2)(5c^3 - 4ce + 4d^2) + (c^2 + e)(2df - e^2) - cf^2, \\ J &= \text{the invariant of the fourth degree in the coefficients,} \end{aligned}$$

and $K =$ the invariant of the eighth degree.

(*See* Am. J. of M., Vol. XXIII, pp. 49 and 56.)

The quintic is isodyadic if

$$y_1 y_4 = y_3 y_2 \neq 0, \tag{4}$$

and $\therefore$ if

$$c \neq 0 \qquad \text{and} \qquad C = 0. \tag{5}$$

$cC = 0$ may be arranged in the form

$$\{cf - d(c^2 + e)\}^2 - (c^3 - ce + d^2)\{(5c^2 - e)^2 + 16cd^2\} = 0. \tag{6}$$

Substituting $-p$ for c, $-p\alpha$ for $2d$, $-p\beta$ for e and $-p\gamma$ for f, this becomes

$$\{2\gamma - \alpha(\beta - p)\}^2 - \{\alpha^2 - 4(\beta + p)\}\{(\beta + 5p)^2 - 4p\alpha^2\} = 0. \tag{7}$$

Let, now,

$$(\beta + 5p)^2 - 4p\alpha^2 = \mu^2\{\alpha^2 - 4(\beta + p)\}; \tag{8}$$

then will

$$2\gamma - \alpha(\beta - p) = \mu\{4(\beta + p) - \alpha^2\} \tag{9}$$

and (8) may be written

$$(\beta + 5p + 2\mu^2)^2 = (4p + \mu^2)(\alpha^2 + 4\mu^2). \tag{10}$$

If now

$$4p + \mu^2 = \lambda^2(\alpha^2 + 4\mu^2) \tag{11}$$

(10) becomes

$$\beta + 5p + 2\mu^2 = \lambda(\alpha^2 + 4\mu^2). \tag{12}$$

Hence, α, λ, μ being any given numbers, if

$$4p = \lambda^2(4\mu^2 + \alpha^2) - \mu^2, \tag{11$'$}$$

$$\beta = \lambda(\alpha^2 + 4\mu^2) - 5p - 2\mu^2, \tag{12$'$}$$

$$2\gamma = \beta(\alpha + 4\mu) - p(\alpha - 4\mu) - \alpha^2\mu, \tag{9$'$}$$

the quintic

$$x^5 - 5p(2x^3 + \alpha x^2 + \beta x) - p\gamma = 0 \tag{13}$$

will be isodyadic.

Examples.—1. If $\alpha = -19$, $\lambda = \frac{9}{13}$, $\mu = 11$, the quintic is

$$x^5 - 355(2x^3 - 19x^2 - 12x) - 7171 = 0.$$

2. If $\alpha = -19$, $\lambda = -\frac{9}{13}$, $\mu = 11$, the quintic is

$$x^5 - 355(2x^3 - 19x^2 - 1182x) + 1031204 = 0.$$

(The former of these is the Gaussian of the quintic section of $x^{71} - 1 = 0$; the latter is its quadratic conjugate.)

3. If $\alpha = -29$, $\lambda = \frac{57}{109}$, $\mu = 121$, the quintic is

$$x^5 - 2005(2x^3 - 29x^2 - 222x) - 589871 = 0.$$

4. If $\alpha = -29$, $\lambda = -\frac{57}{109}$, $\mu = 121$, the quintic is

$$x^5 - 2005(2x^3 - 29x^2 - 62352x) + 4404222699 = 0.$$

(Ex. 3 is the Gaussian of the quintic section of $x^{401} - 1$; Ex. 4 is its quadratic conjugate.)

Solution.—If the coefficients of the quintic

$$x^5 + 10cx^3 + 10dx^2 + 5ex + f = 0 \tag{1}$$

satisfy the relations $c \neq 0$ and

$$(5c^3 - 3ce + 4d^2)(5c^3 - 4ce + 4d^2) + (c^2 + e)(2df - e^2) - cf^2 = 0,$$

it is isodyadic and may be solved as follows.

Write (1) in the form

$$x^5 - 5p(2x^3 + \alpha x^2 + \beta x) - p\gamma = 0; \tag{13}$$

then will the roots be

$$x_n = \omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4,$$

in which

$$(\omega^5 - 1)/(\omega - 1) = 0, \qquad n = 1, 2, 3, 4, 5;$$

and

$$y_1 y_4 = y_3 y_2 = p, \tag{i}$$
$$y_1^2 y_3 + y_4^2 y_2 + y_3^2 y_4 + y_2^2 y_1 = p\alpha, \tag{ii}$$
$$y_1^3 y_2 + y_4^3 y_3 + y_3^3 y_1 + y_2^3 y_4 = p(p+\beta), \tag{iii}$$
$$(y_1^5 + y_4^5 + y_3^5 + y_2^5 = p\gamma). \tag{iv}$$

For convenience of notation write

$$\rho = \alpha^2 - 4(p+\beta), \tag{v}$$
$$\sigma = \alpha + \sqrt{\rho},$$
$$\tau = \alpha - \sqrt{\rho},$$
$$\mu = \{\alpha(\beta - p) - 2\gamma\}/\rho,$$

$\therefore$
$$\sigma\tau = 4(p+\beta)$$

and
$$\sqrt{\{(\sigma^2 - 16p)(\tau^2 - 16p)\}} = 4\mu\sqrt{\rho}.$$

From $\sqrt{\{(\text{ii})^2 - 4(\text{iii})\}}$ and (v)

$$y_1^2 y_3 + y_4^2 y_2 - y_3^2 y_4 - y_2^2 y_1 = p\sqrt{\rho};$$

$\therefore$ $y_1^2 y_3 + y_4^2 y_2 = \frac{1}{2}p\sigma$

and $y_3^2 y_4 + y_2^2 y_1 = \frac{1}{2}p\tau;$

$\therefore$ $y_1^2 y_3 - y_4^2 y_2 = \frac{1}{2}p\sqrt{(\sigma^2 - 16p)}$

and $y_3^2 y_4 - y_2^2 y_1 = \frac{1}{2}p\sqrt{(\tau^2 - 16p)};$

$\therefore$
$$y_1^2 y_3 = \tfrac{1}{4}p\{\sigma + \sqrt{(\sigma^2 - 16p)}\},$$
$$y_4^2 y_2 = \tfrac{1}{4}p\{\sigma - \sqrt{(\sigma^2 - 16p)}\},$$
$$y_3^2 y_4 = \tfrac{1}{4}p\{\tau + \sqrt{(\tau^2 - 16p)}\},$$
$$y_2^2 y_1 = \tfrac{1}{4}p\{\tau - \sqrt{(\tau^2 - 16p)}\},$$

$\therefore$
$$\begin{aligned}
y_1^5 &= \tfrac{1}{64}p\{\sigma + \sqrt{(\sigma^2 - 16p)}\}^2\{\tau - \sqrt{(\tau^2 - 16p)}\} \\
&= \tfrac{1}{8}p\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho} \\
&\qquad + \sqrt{[\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\}, \\
y_4^5 &= \tfrac{1}{8}p\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho} \\
&\qquad - \sqrt{[\{2\gamma + (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\}, \\
y_3^5 &= \tfrac{1}{8}p\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho} \\
&\qquad + \sqrt{[\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\}, \\
y_2^5 &= \tfrac{1}{8}p\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho} \\
&\qquad - \sqrt{[\{2\gamma - (3p + \beta - \alpha\mu)\sqrt{\rho}\}^2 - 64p^3]}\}.
\end{aligned}$$

Expressed in the coefficients of (1) these become

$$y_1^5 = -\tfrac{1}{4}[f + R - \sqrt{\{(f+R)^2 + 16c^5\}}],$$
$$y_4^5 = -\tfrac{1}{4}[f + R + \sqrt{\{(f+R)^2 + 16c^5\}}],$$
$$y_3^5 = -\tfrac{1}{4}[f - R - \sqrt{\{(f-R)^2 + 16c^5\}}],$$
$$y_2^5 = -\tfrac{1}{4}[f - R + \sqrt{\{(f-R)^2 + 16c^5\}}],$$

in which

$$R = \frac{4(c^2 - e)(3c^2 - e) + cd^2 + df}{\sqrt{\{16c(c^2 - e) + d^2\}}}.$$

Examples.—1.

$$x^5 - 55(2x^3 + x^2 - 42x) + 979 = 0.$$

Here, $p = 11$, $\alpha = 1$, $\beta = -42$, $\gamma = -89$ and $\therefore \rho = 125$, $\mu = 1$,

$$y_1y_4 = y_3y_2 = 11,$$
$$y_1^2y_3 + y_4^2y_2 + y_3^2y_4 + y_2^2y_1 = 11,$$
$$y_1^3y_2 + y_4^3y_3 + y_3^3y_1 + y_2^3y_4 = -341;$$
$$\therefore \quad y_1^2y_3 + y_4^2y_2 = \tfrac{11}{2}(1 + 5\sqrt{5}),$$
$$y_3^2y_4 + y_2^2y_1 = \tfrac{11}{2}(1 - 5\sqrt{5});$$
$$\therefore \quad y_1^2y_3 = \tfrac{11}{4}\{1 + 5\sqrt{5} + \sqrt{(-50 + 10\sqrt{5})}\},$$
$$y_2^2y_1 = \tfrac{11}{4}\{1 - 5\sqrt{5} - \sqrt{(-50 - 10\sqrt{5})}\};$$
$$\therefore \quad y_1^5 = -\tfrac{11}{4}[89 + 25\sqrt{5} - \sqrt{\{(89 + 25\sqrt{5})^2 - 16 \times 11^3\}}].$$

(This is the cyclotomic quintic for $x^{11} - 1 = 0$.)

2. The quadratic conjugate of Ex. 1 is

$$x^5 - 55(2x^3 + x^2 - 72x) + 1804 = 0$$

with $\quad y_1 = -11[41 + 25\sqrt{5} - \sqrt{\{(41 + 25\sqrt{5})^2 - 11^3\}}]$.

3. $\quad x^5 - 800(2x^3 + 5x^2 - 534x) - 160 \times 2828 = 0.$

Here $\quad \rho = 39^2 \quad$ and $\quad \mu = -6$,

$$y_1^5 = 1600(59 + 31i\sqrt{39}),$$
$$y_3^5 = 1280(103 + 37i\sqrt{39}).$$

A. If (13) be cyclotomic and p be a prime number $\equiv 1 \bmod 10$, α will be $\equiv 1 \bmod 5$ and $5\{(\beta + 5p)^2 - 4p\alpha^2\}$ will be a rational square (AM. J. M., Vol. XXI, p. 272). Hence,

If p is a prime $\equiv 1 \bmod 10$, $4p = u^2 - 5v^2$ is solvable in integers.

B. In general (AM. J. M., Vol. XXI, p. 277), if n and p are primes, $n \equiv 1 \bmod 4$, $p \equiv 1 \bmod 2n$, $4p = u^2 - nv^2$ is solvable in integers. Hence, also, if $n \equiv 1 \bmod 8$, $p = u^2 - nv^2$ is solvable, but if $n \equiv 5 \bmod 8$, $p = u^2 - 5v^2$ is solvable only if $4 = u^2 - 5v^2$ is solvable; e.g., $p = u^2 - 5v^2$ has no integral solutions if $n \equiv 1 \bmod 4(2m + 3)^2$. Ex. $773 = 7205048979^2 - 193 \times 518630774^2$ and $4 \times 2153 = 71821^2 - 269 \times 4379^2$.

ON CLASS NUMBER RELATIONS FOR BILINEAR FORMS IN FOUR VARIABLES.

By E. T. Bell.

1. In the notation of Kronecker,* $\overline{Cl}(\Delta)$ is the number of classes of bilinear forms

$$Ax_1y_1 + Bx_1y_2 - Cx_2y_1 + Dx_2y_2, \qquad \Delta \equiv AD + BC,$$

of determinant Δ, in which at least one of the extreme coefficients A, D is odd and $B + C$ is even. He shows (loc. cit., p. 494) that

$$\sum_{n=1}^{\infty} \overline{Cl}(n)q^n = \frac{6k^2K^2}{\pi^2 \sqrt[4]{q}} \int_0^1 H((2v+1)K) \sin^2 am2vK \cos v\pi dv,$$

in the notation of the *Fundamenta Nova.* In other words that part of the Fourier cosine series for the function

$$\frac{6k^2K^2}{\pi^2 \sqrt[4]{q}} H((2v+1)K) \operatorname{sn}^2 2vK \cos v\pi,$$

which is independent of the argument v is $\sum \overline{Cl}(n)q^n$. The part containing v is not obtained, and is in fact neglected in the entire discussion. Attending to this part we find several interesting class number relations for binary quadratic forms and, as the simplest consequence of the whole expansion, an elegant relation of a similar kind for $\overline{Cl}(n)$. Here we shall restrict the discussion to this relation, believed to be the first of its kind. We recall that Kronecker obtained $\overline{Cl}(n)$ in the form (loc. cit., p. 452) $12\sum \mathrm{F}(n - h^2)$, the $\sum$ referring to all integers $h \gtreqless 0$ such that $n - h^2 > 0$, $\mathrm{F}(n)$ being his class number function† for binary quadratic forms of negative determinant $-n$. This will afford a check on one phase of the following developments.

2. Changing Kronecker's notation to the small thetas of Jacobi, we are to expand $\psi(v)$ in a cosine series, where

$$\sqrt[4]{q}\psi(v) \equiv \frac{3}{2} \cos v\vartheta_2^2\vartheta_3^2 \frac{\vartheta_1^2(v)\vartheta_2(v)}{\vartheta_0^2(v)}.$$

Denote by $\psi_0(v)$, $\psi_1(v)$ respectively the parts of $\psi(v)$ independent of v, not independent of v. Then $\psi_0(v) = \sum \overline{Cl}(n)q^n$. To obtain $\psi_1(v)$ we introduce $\chi(v)$, defined by

$$\chi(v) \equiv \vartheta_2^2\vartheta_3 \frac{\vartheta_1^2(v)\vartheta_2(v)}{\vartheta_0^2(v)},$$

* Werke, Vol. 2, pp. 425–495. See also Dickson's "History," Vol. 3, pp. 129–130, 286.

† For the distinction between F and the more usual F see Dickson's "History," Vol. 3, p. 109; or Kronecker, loc. cit., p. 449.

which, in another notation, is one of the functions discussed by G. Humbert.* The reduced form of $\chi(v)$ suitable for this kind of work was given in a former paper.† We found

$$\chi(v) = 2\sum q^{\alpha/4}\left[2\sum F(\alpha - b^2)\cos bx - \sum{}'(\delta - d)\cos\left(\frac{\delta + d}{2}\right)v\right],$$

in which the outer $\sum$ refers to all positive integers $\alpha \equiv 1 \bmod 4$, the coefficient of $q^{\alpha/4}$ is in $\square$, the inner $\sum$ refers to all odd integers $b \gtrless 0$ which render $\alpha - b^2 > 0$, $F(n)$ is the class number for binary quadratic forms of determinant $-n$ with the usual conventions, and $\sum'$ refers to all positive divisors d, δ of α such that $\alpha = d\delta$, $d > \sqrt{\alpha}$. We have

$$\sqrt[4]{q}\psi(v) = \tfrac{3}{2}\cos v \cdot \vartheta_3 \cdot \chi(v).$$

Writing $c = |b|$ we find

$$\chi(v)\cos v = 4\sum q^{\alpha/4}[\sum F(\alpha - c^2)\{\cos(c+1)v + \cos(c-1)v\}]$$

$$+ \sum q^{\alpha/4}\left[\sum{}'(d - \delta)\left\{\cos\left(\frac{\delta + d}{2} + 1\right)v + \cos\left(\frac{\delta + d}{2} - 1\right)v\right\}\right],$$

the sum with respect to c extending to all $c = 1, 3, 5, \cdots$, such that

$$\alpha - c^2 > 0.$$

Calculating the part of $\chi(v)\cos v$ independent of v, we note that zero arguments can appear only in terms of the form $\cos(c - 1)v$, $\cos\left(\frac{\delta + d}{2} - 1\right)v$. From the first, $c = 1$; from the second $\delta + d = 2$, and hence $\delta = d = 1$. But the restriction $d < \sqrt{\alpha}$ imposed by $\sum'$ excludes the second. The part of $\chi(v)\cos v$ independent of v is therefore $4\sum q^{\alpha/4}F(\alpha - 1)$. Hence from the definitions of the functions,

$$\psi_0(v) = 6\sum q^{(\alpha-1)/4}F(\alpha - 1) \times \sum q^{n^2},$$

the second $\sum$ referring to $n = 0, \pm 1, \pm 2, \cdots$. Recalling that $\alpha \equiv 1 \bmod 4$, and that $F(4n) = 2F(n)$ for all integers $n \geqq 0$, we now replace α by $4n + 1$ and get

$$\psi_0(v) = 12\sum q^n[\sum F(n - h^2)] = \sum q^n \overline{C}l(n),$$

the sum with respect to h referring to all $h = 0, \pm 1, \pm 2, \cdots$, such that $n - h^2 > 0$. This checks $\psi_0(v)$.

3. The part of $\chi(v)\cos v$ not independent of v can now be written (on

* *Journal des Mathématiques*, 6 Série, Vol. 3 (1907), pp. 337–449. Humbert's entire set of 24 expansions, ibid., pp. 349–354, can be used to derive bilinear class number relations.

† *Quarterly Journal*, Vol. 59, No. 196 (1923), pp. 322–337, where the entire set of 24 is reduced.

omitting the term $\cos (c-1)v$ for $c=1$) in the form

$$4\sum q^{\alpha/4}[\sum\{F(\alpha-\overline{s-1}^2)+F(\alpha-\overline{s+1}^2)\}\cos sx]$$
$$-\sum q^{\alpha/4}\left[\sum'(\delta-d)\left\{\cos\left(\frac{\delta+d}{2}+1\right)v+\cos\left(\frac{\delta+d}{2}-1\right)v\right\}\right],$$

the inner $\sum$ in the first referring to all even integers $s>0$ such that no F occurs with a zero or negative argument. Hence, multiplying $\chi(v)\cos v$ by ϑ_3, dividing the result by $\sqrt[4]{q}$, and making some obvious simplifications in the resulting series, we finally obtain

$$\begin{aligned}\psi(v)=\sum q^n[\overline{Cl}(n)&+24\sum\sum\{\mathrm{F}(n-t_1^2-t^2+t)\\&+\mathrm{F}(n-t_1^2-t^2-t)\}\cos 2tv]\\&-\frac{3}{2}\sum q^n\left[\sum'(\delta_1-d_1)\left\{\cos\left(\frac{\delta_1+d_1}{2}+1\right)v\right.\right.\\&\left.\left.\qquad+\cos\left(\frac{\delta_1+d_1}{2}-1\right)v\right\}\right],\end{aligned}$$

in which the summations are as follows: the outer $\sum$'s refer to $n=1, 2, 3, \cdots$; the $\sum\sum$ to all $t_1 \gtreqless 0$ and to all $t>0$ such that the argument of no F is zero or negative; the $\sum'$ to all divisors d_1, δ_1 of $\alpha_1=d_1\delta_1$ such that $d_1<\sqrt{\alpha_1}$, and α_1 is determined from *all* solutions of $0<4n+1-4t_1^2=\alpha_1$ (n fixed, t_1 as above).

Note that the second sum contains no term independent of v. The first of the sums in $\psi(v)$ can be materially simplified, at least in expression. For a moment write $X(n)=24\sum\sum\mathrm{F}(n-t_1^2-t^2+t)$, this being one of the double sums above. Then, the summation referring to all values of $n_1=1, 2, 3, \cdots$, and of n, t_1, t as defined, we have

$$\begin{aligned}\sum X(n)q^n&\equiv 24\sum q^n[\sum\sum\mathrm{F}(n-t_1^2-t^2+t)],\\&=24\sum q^{n_1}\mathrm{F}(n_1)\times\sum q^{t_1^2}\times\sum q^{t^2-t},\\&=24\sum q^n[\sum\mathrm{F}(n-t_1^2)]\times\sum q^{t^2-t},\\&=2\sum q^n[\sum\overline{Cl}(n-t^2+t)],\end{aligned}$$

the inner $\sum$ referring to all $t>0$ such that $n-t^2+t>0$. Hence

$$\begin{aligned}\psi(v)=\sum q^n[\overline{Cl}(n)&+2\sum\overline{Cl}(n-a^2+a)\cos 2av]\\&-\frac{3}{2}\sum q^n\left[\sum'(\delta_1-d_1)\left\{\cos\left(\frac{\delta_1+d_1}{2}+1\right)v\right.\right.\\&\left.\left.\qquad+\cos\left(\frac{\delta_1+d_1}{2}-1\right)v\right\}\right],\end{aligned}$$

the $\sum$ with respect to a referring to all $a=\pm 1, \pm 2, \pm 3, \cdots$ such that $n-a^2+a>0$. The rest of the letters are as before.

4. As the simplest consequence of this expansion we note that $\psi(0) = 0$, since $\vartheta_1(0) = 0$. Hence

$$\overline{Cl}(n) + 2\sum\overline{Cl}(n - a^2 + a) = 3\sum{}'(\delta_1 - d_1),$$

the notation being as before. Denote by $\lambda(n)$ the excess of the sum of all those divisors of n that exceed $\sqrt{n}$ over the sum of all those that are exceeded by $\sqrt{n}$. Then $\sum{}'(\delta_1 - d_1) = \lambda(\alpha_1)$, and hence the right of the foregoing equation is $3\sum\lambda(4n + 1 - 4t^2)$, from the definition of α_1. Finally then we obtain the bilinear class number relation

$$\overline{Cl}(n) + 2\sum\overline{Cl}(n - a^2 + a) = 3[\lambda(4n + 1) + \sum\lambda(4n + 1 - 4a^2)],$$

the summations referring to all $a = \pm 1, \pm 2, \pm 3, \cdots$ that render the arguments of the function > 0.

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RELATIVE INCLUSIVENESS OF CERTAIN DEFINITIONS OF SUMMABILITY.

By David Sherman Morse.

1. Introduction.

The number of definitions that have been given for evaluating a divergent series is so large that the study of the relationships existing between various definitions has become of importance. It is the purpose of this paper to contribute to this study by considering certain definitions from the standpoint of relative inclusiveness.

The terminology used in the literature of the subject differs somewhat with different writers. Because of this, it seems desirable to define some of the terms we are going to use.* A definition is said to be *regular* if it evaluates every convergent sequence, giving to it the value to which it converges. A definition, A, is said to *include* another, B, provided every sequence summable B is summable A to the same value. Two definitions are said to be *equivalent* if each includes the other. Two definitions are said to be *mutually consistent* if, whenever each of them evaluates a sequence, the two values are the same.

We shall have occasion to refer to several definitions that assign sums to divergent series. We now list some of these definitions, together with the symbols by which we refer to them. Unless otherwise stated, we use the symbol

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$$

and define

$$x_n = u_1 + u_2 + \cdots + u_n.$$

Cesàro's rth Mean, C_r.†

This definition assigns to a divergent series the value $\lim_{n\to\infty} y_n$, if this limit exists, where

$$y_n = \frac{S_n^{(r)}}{A_n^{(r)}};$$

* The terminology here used is that of Prof. Hurwitz's paper, *Report on topics in the theory of divergent series;* Bulletin of the American Mathematical Society, Vol. 28 (1922), p. 17. This paper will be referred to as *Report.*

† *Bulletin des Sciences Mathématiques*, Ser. 2, Vol. 14 (1880), p. 114.

with

$$S_n^{(r)} = \frac{r(r+1)\cdots(r+n-2)}{(n-1)!}x_1 + \frac{r(r+1)\cdots(r+n-3)}{(n-2)!}x_2 + \cdots + rx_{n-1} + x_n,$$

and

$$A_n^{(r)} = 1 + r + \frac{r(r+1)}{2!} + \cdots + \frac{r(r+1)\cdots(r+n-2)}{(n-1)!}.$$

An equivalent expression for y_n is:

$$y_n = \frac{n(n+1)\cdots(n+r-2)x_1 + \cdots + (r-1)!\,x_n}{n(n+1)\cdots(n+r-1)/r}.$$

*LeRoy's Definition.**

The generalized value for a divergent series according to this definition is $\lim_{t\to 1-} y(t)$ where

$$y(t) = \sum_{n=1}^{\infty} \frac{\Gamma(\overline{n-1}\,t+1)}{(n-1)!}u_n, \qquad 0 < t < 1.$$

Borel's Integral Definition.†

Form the function

$$U(t) = u_1 + u_2t + u_3t^2/2! + \cdots;$$

if $U(t)$ converges for all t, and if

$$\int_0^\infty e^{-t}U(t)dt$$

exists and is equal to l, then the series is said to be summable Borel to the value l.

The Exponential Mean, E_r.‡

The generalized value is $\lim_{n\to\infty} y_n$, if this limit exists, where

$$y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!}\,\frac{(r-1)^{n-k}}{r^{n-1}}x_k.$$

The Riesz Means, (R_λ, k).§

Form

$$y(t) = t^{-k}\sum_{\lambda_n<t}(t-\lambda_n)^k u_n,$$

* *Annales Fac. Sci. Toulouse*, Ser. 2, Vol. 2 (1900), p. 327.

† Borel: *Leçons sur les séries divergentes* (Paris, 1901).

‡ Hausdorff: *Mathematische Zeitschrift*, Vol. 9 (1921), p. 86. Also, Hurwitz: *Report*.

§ *Comptes Rendus*, Vol. 149 (1909), p. 910.

in which λ_n is a sequence of positive real increasing numbers whose limit is infinite. Then the generalized value of the series is given by

$$\lim_{t \to \infty} y(t).$$

The Dirichlet's Series Definitions, D_{λ_n}.

Hardy* gives a definition which he states is "substantially equivalent" to LeRoy's and has, in the case of a power series of finite radius of convergence, "precisely similar powers." This definition is†

$$\lim_{\delta \to 0} \sum_{\nu=0}^{\infty} u_\nu e^{-\delta\nu \log \nu}.$$

This definition leads us to a set of definitions which we shall call the Dirichlet's series definitions.

We define as the generalized value of the series, $\sum_{n=1}^{\infty} u_n$,

$$\lim_{\substack{t \to t_0 \\ T}} \sum_{n=1}^{\infty} u_n e^{-\lambda_n t},$$

where T is a point set, in the real or complex plane, having a limit point t_0, not belonging to the set, and where λ_n is a sequence of positive increasing real numbers whose limit is infinite.

A special case of the Dirichlet's series definitions is well known; namely, the case where $\lambda_n = n$, $t_0 = 0$, and T is a point set along the axis of reals, $0 < t$. This gives

$$\lim_{t \to 0} \sum_{n=1}^{\infty} u_n e^{-nt} = \lim_{t \to 0} e^{-t} \sum_{n=1}^{\infty} u_n (e^{-t})^{n-1} = \lim_{t \to 0} \sum_{n=1}^{\infty} u_n (e^{-t})^{n-1}.$$

If we replace e^{-t} by x, we get

The Euler Power Series Definition.

The generalized value is given by

$$\lim_{x \to 1-} \sum_{n=1}^{\infty} u_n x^{n-1}.$$

We wish to study another special case of this set of definitions which differs from Hardy's definition only to the extent that we write $\sum_{n=1}^{\infty} u_n$ while Hardy wrote $\sum_{n=0}^{\infty} u_n$. This definition, we call

* *Quarterly Journal of Mathematics*, Vol. 42 (1911), p. 193.

† $\delta \to \infty$, in above reference, is evidently a typographical error.

The Dirichlet's Series Definition $D_{n \log n}$.

$$\lim_{\substack{t \to 0 \\ T}} \sum_{n=1}^{\infty} u_n e^{-nt \log n}.$$

This definition is regular* when T is a set of points along the axis of reals. Sufficient conditions that a definition of the kind be regular are given by Carmichael.† They are, in this case,

1. $\lim_{t \to 0} e^{-nt \log n} = 1$.

2. $\sum_{n=1}^{\infty} |e^{-nt \log n} - e^{-(n+1)t \log (n+1)}|$ converges for each $t > 0$ and is bounded for all $t > 0$.

The first condition requires no proof. The second condition is satisfied also. Since $n \log n$ is an increasing function for $n > 1/e$,

$$e^{-nt \log n} - e^{-(n+1)t \log (n+1)} > 0$$

for $n > 1/e$, and the sum in condition 2 can be replaced by

$$\sum_{n=1}^{\infty} [e^{-nt \log n} - e^{-(n+1)t \log (n+1)}].$$

Since $\lim_{n \to \infty} e^{-nt \log n} = 0$ for $t > 0$, it follows that

$$\sum_{n=1}^{\infty} [e^{-nt \log n} - e^{-(n+1)t \log (n+1)}] = 1$$

and the second condition is satisfied.

In section 2 of this paper, we prove that the Dirichlet's series definition, $D_{n \log n}$, includes C_r. Then, in section 3, we consider this definition, $D_{n \log n}$, from a Dirichlet's series standpoint, giving a proof of regularity when $t \to 0$ over other point sets than the axis of reals. We also give another proof that $D_{n \log n}$ includes C_r.

There are, in the literature, at least two misleading statements regarding the relationship of the LeRoy and Borel definitions. Hardy, in the *Quarterly Journal of Mathematics*, Vol. 35, p. 37, states the following conclusion to a proof. "It follows that if $u_0 + u_1 + u_2 + \cdots$ is summable by the exponential method, it is summable also by M. LeRoy's method, and that the two sums are the same." This seems to imply that LeRoy's definition includes Borel's. Upon looking up a reference that the author makes to one of his earlier papers in order to justify a step in the proof, it becomes evident that he has proved the theorem only for the case where the series $u_0 + u_1 + u_2 + \cdots$ is convergent and has, consequently, proved that

* The definition is also *totally regular*. See *Report*, p. 30.

† *Bulletin of the American Mathematical Society*, Vol. 25 (1918–19), p. 120. These conditions are, in fact, necessary.

LeRoy's definition is regular provided Borel's is regular. If the section, in which the proof occurs, is read from the beginning, this is seen to be the purpose of the author. Bromwich also has made a misleading statement in this connection. In his "Theory of Infinite Series," page 299, he proves that, if a series is summable Borel, it is then summable LeRoy to the same value *provided* the series in the LeRoy definition converges absolutely. At the end of the proof, he forgets, evidently, his assumption of absolute convergence, and states: "Hence LeRoy's definition coincides with Borel's, whenever the latter is convergent."

In section 4, we show that Borel's definition is not included in LeRoy's. We prove that, if a series is summable Borel, then it is summable LeRoy to the same value provided the LeRoy series converges; and give a sufficient condition that a series summable Borel shall be summable LeRoy to the same value. From this condition, we get several theorems, of which the following is typical: *If a series is summable C_r and Borel, it is summable LeRoy to the same value.* We also prove, in this section, that the Dirichlet's series definition $D_{n \log n}$ does not include Borel's definition and that Borel's definition does not include $D_{n \log n}$.

In section 5, we have indicated how far we have been able to go towards proving that LeRoy's definition includes C_r and have pointed out why we believe that LeRoy's definition does include C_r.

2. THE RELATION BETWEEN THE DIRICHLET'S SERIES DEFINITION $D_{n \log n}$ AND C_r.

Bromwich* has proved that if a series, $\sum_{n=1}^{\infty} u_n$, is summable C_r to the sum l and if $f_n(t)$ is a function of t such that

1. $\lim_{t \to 0} f_n(t) = 1$,
2. $\lim_{n \to \infty} n^r f_n(t) = 0, \quad t > 0$,
3. $\sum_{n=1}^{\infty} n^r \left| \Delta^{r+1} f_n(t) \right|$ converges for each $t > 0$ and is bounded for all $t > 0$;

then $\sum_{n=1}^{\infty} f_n(t) u_n$ converges, $t > 0$, and

$$\lim_{t \to 0} \sum_{n=1}^{\infty} f_n(t) u_n = l.\dagger$$

Hurwitz‡ has shown that (2) may be replaced by the more general condition

* *Mathematische Annalen*, Vol. 65 (1907–08), p. 359.

† Compare form of statement used by C. N. Moore, *Bulletin of American Mathematical Society*, Vol. 25, page 267.

‡ Abstract, *Bulletin of American Mathematical Society*, Vol. 28 (1922), page 156.

2′. for each $t > 0$, $n^r f_n(t)$ is bounded for all n (the bound may depend on t);

and that conditions (1), (2′), and (3) are necessary if the result is to hold for every series summable C_r.

We make use of this theorem to prove

THEOREM I. *The Dirichlet's series definition, $D_{n \log n}$, includes C_r for all positive integral values of r.*

We assume that $\sum_{n=1}^{\infty} u_n$ is summable C_r to the value l and desire to prove that

$$\sum_{n=1}^{\infty} e^{-nt \log n} u_n$$

converges, $t > 0$, and that

$$\lim_{t \to 0} \sum_{n=1}^{\infty} e^{-nt \log n} u_n = l.$$

From Bromwich's theorem, this will be true provided

1. $\lim_{t \to 0} e^{-nt \log n} = 1$,
2. $\lim_{t \to 0} n^r e^{-nt \log n} = 0, \quad t > 0$,
3. $\sum_{n=1}^{\infty} n^r |\Delta^{r+1} e^{-nt \log n}|$ converges for each $t > 0$ and is bounded for all $t > 0$.

Before showing that these three conditions are satisfied, we give four lemmas.

LEMMA 1. *If*

1. $\sum_{k=p}^{q} F(k, t)$ *is bounded for all p, q and all $t > 0$,*
2. *the number of changes of sign (with varying k) of $F(k, t)$ is finite for each constant $t > 0$ and bounded for all $t > 0$,*

then

$$\sum_{k=1}^{\infty} |F(k, t)| \text{ converges for each } t > 0 \text{ and is bounded for all } t > 0.$$

In this lemma, condition 1 may be replaced by the equivalent condition

1′. $\sum_{k=1}^{q} F(k, t)$ is bounded for all q and all $t > 0$.

For each t there will exist an index $k = g(t)$ such that for $k > g(t)$, $F(k, t)$ has the same sign. Form $\sum_{k=1}^{n} |F(k, t)|$, where $n > g(t)$, and break this sum into partial sums each consisting of terms in which $F(k, t)$ retains

one sign. Then

$$\sum_{k=1}^{n}|F(k,t)| = \sum_{(1)}|F(k,t)| + \sum_{(2)}|F(k,t)| + \cdots + \sum_{(n)}|F(k,t)|$$
$$= \left|\sum_{(1)}F(k,t)\right| + \left|\sum_{(2)}F(k,t)\right| + \cdots + \left|\sum_{(n)}F(k,t)\right|.$$

By condition 1, each group is less than a constant A, independent of t; and by condition 2, the number of groups is less than a constant M, independent of t; hence

$$\sum_{k=1}^{n}|F(k,t)| < M\cdot A;$$

from which the conclusion of the theorem follows.

LEMMA 2. *If* $\varphi(x) = \dfrac{f(x)}{x}$, *then the rth derivative of* $\varphi(x)$ *is given by*

$$\varphi^{(r)}(x) = \frac{\sum_{k=0}^{r}(-1)^k \dfrac{r!}{(r-k)!}x^{r-k}f^{(r-k)}(x)}{x^{r+1}}.$$

LEMMA 3. *If* $\varphi(n)$ *possesses r derivatives, then*

$$(-1)^r\Delta^r\varphi(n) = \varphi^{(r)}[n+\theta_r], \qquad 0<\theta_r<r.$$

LEMMA 4. *If* $f(x) = e^{-tx\log x}$, *then the rth derivative of* $f(x)$ *is given by*

$$f^{(r)}(x) = (-t)^r f(x)\left[(1+\log x)^r - \frac{r(r-1)}{2}\frac{(1+\log x)^{r-2}}{tx}\frac{\varphi_{r-3}(\log x)}{t^2x^2} + \cdots + \frac{\varphi_1(\log x)}{t^{r-2}x^{r-2}} + \frac{(r-2)!}{t^{r-1}x^{r-1}}\right],$$

where $\varphi_k(\log x)$ represents a polynomial in $\log x$ of degree $\leq k$.

This lemma can be established by mathematical induction.

We are now in a position to prove that the function $e^{-nt\log n}$ satisfies the three conditions of Bromwich's theorem.

The first condition that $\lim\limits_{t\to 0} e^{-nt\log n} = 1$ is satisfied.

In order to show that the second condition is satisfied, we write

$$n^r e^{-nt\log n} = n^{r-nt}\to 0 \qquad \text{as} \qquad n\to\infty,\quad t>0.$$

It remains to prove that $\sum\limits_{n=1}^{\infty} n^r|\Delta^{r+1}e^{-nt\log n}|$ converges for each $t>0$ and is bounded for all $t>0$. If n^r in this expression is replaced by any polynomial of the rth degree in n, the new expression will surely converge and be bounded, or fail to converge and be bounded, under the same conditions as the original expression. We, therefore, choose to show that

$$\sum_{n=1}^{\infty}\frac{(n+r)!}{n!}|\Delta^{r+1}e^{-nt\log n}|$$

converges for each $t > 0$ and is bounded for all $t > 0$. We do this by showing that the conditions of Lemma 1 are satisfied. We show that condition 1′ of the lemma is satisfied by computing

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(k+r)!}{k!}\Delta^{r+1}f_k(t) &= \sum_{k=1}^{n-1}\frac{(k+r)!}{k!}f_k(t)\\ &\quad-\sum_{k=1}^{n-1}(r+1)\frac{(k+r)!}{k!}f_{k+1}(t)+\sum_{k=1}^{n-1}\frac{(r+1)r}{2!}\frac{(k+r)!}{k!}f_{k+2}(t)\\ &\quad-\cdots(-1)^{r+1}\sum_{k=1}^{n-1}\frac{(r+1)r}{2!}\frac{(k+r)!}{k!}f_{k+r-1}(t)\\ &\quad+(-1)^{r}\sum_{k=1}^{n-1}(r+1)\frac{(k+r)!}{k!}f_{k+r}(t)\\ &\qquad\qquad+(-1)^{r+1}\sum_{k=1}^{n-1}\frac{(k+r)!}{k!}f_{k+r+1}(t),\end{aligned}$$

where $f_k(t)$ is written for $e^{-kt\log k}$.

We can write

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(k+r)!}{k!}\Delta^{r+1}f_k(t) &= \sum_{k=1}^{n-1}\frac{(k+r)!}{k!}f_k(t)\\ &\quad-\sum_{k=2}^{n}(r+1)\frac{(k+r-1)!}{(k-1)!}f_k(t)+\sum_{k=3}^{n+1}\frac{(r+1)r}{2!}\frac{(k+r-2)!}{(k-2)!}f_k(t)\\ &\quad-\cdots(-1)^{r+1}\sum_{k=r}^{n+r-2}\frac{(r+1)r}{2!}\frac{(k+1)!}{(k-r+1)!}f_k(t)\\ &\quad+(-1)^{r}\sum_{k=r+1}^{n+r-1}(r+1)\frac{k!}{(k-r)!}f_k(t)+(-1)^{r+1}\sum_{k=r+2}^{n+r}\frac{(k-1)!}{(k-r-1)!}f_k(t).\end{aligned}\tag{1}$$

If $n \geq r+3$, we can write part of (1) as follows:

$$\begin{aligned}\sum_{k=r+2}^{n-1}\Bigg[\frac{(k+r)!}{k!}&-(r+1)\frac{(k+r-1)!}{(k-1)!}+\frac{(r+1)r}{2}\frac{(k+r-2)!}{(k-2)!}\\ &-\cdots(-1)^{r}(r+1)\frac{k!}{(k-r)!}+(-1)^{r+1}\frac{(k-1)!}{(k-r-1)!}\Bigg]f_k(t).\end{aligned}\tag{2}$$

The sum of the first $p+1$ terms inside the bracket, where $p+1 \leq r+2$, is given by

$$\begin{aligned}&\frac{(-1)^p}{k}\,\frac{r!\,(k+r-p)!}{p!\,(r-p)!\,(k-p-1)!}\\ &\qquad\qquad=\frac{(-1)^p}{k}\,\frac{r!\,(k+r-p)!\,(r-p+1)}{p!\,(r-p+1)!\,(k-p-1)!},\end{aligned}\tag{3}$$

the first formula failing to have a meaning when $p = r+1$. This formula follows from the usual proof by mathematical induction. There are $r+2$ terms in the brackets. If we put $p = r+1$ in (3), we get zero. Therefore, (2) is zero.

We now show how to calculate the terms of (1) not included in (2). The coefficients of $f_{r+1}(t), f_r(t), \cdots, f_2(t)$ can be found from (3) by replacing k by $r+1, r, \cdots, 2$ and p by $r, r-1, \cdots, 1$, respectively. The coefficient of $f_1(t)$ is $(r+1)!$ Since we do not need to know these coefficients explicitly, we do not compute them.

In order to find the coefficients of $f_n(t), f_{n+1}(t), \cdots, f_{n+r}(t)$, we consider the coefficient of $f_n(t)$. It is

$$(-1)^{r+1}\left[\frac{(n-1)!}{(n-r-1)!}-(r+1)\frac{n!}{(n-r)!}+\frac{(r+1)r}{2!}\frac{(n+1)!}{(n-r+1)!}\right.$$
$$\left.-\cdots(-1)^r(r+1)\frac{(n+r-1)!}{(n-1)!}\right].$$

The sum of $p+1$ terms of this expression, where $p \leq r$, can be shown by mathematical induction to be

$$\frac{(-1)^p}{n}\frac{r!\,(n+p)!}{(r-p)!\,p!\,(n-r+p-1)!},\qquad p \leq r. \tag{4}$$

There are $r+1$ terms in the bracket and the coefficient of $f_n(t)$ is given by (4) to be $(-1)^r\dfrac{(n+r)!}{n!}$.

The coefficients of $f_{n+1}(t), f_{n+2}(t), \cdots, f_{n+r}(t)$ can be found by replacing n by $n+1$, $n+2$, $n+3$, $\cdots$, $n+r$, and p by $r-1$, $r-2$, $\cdots$, 0, respectively, in formula (4).

Finally, we have the result:

$$\begin{aligned}\sum_{k=1}^{r-1}\frac{(k+r)!}{k!}\Delta^{r+1}f_k(t) &= c_1f_1(t)+c_2f_2(t)\\ &+\cdots+c_{r+1}f_{r+1}(t)+(-1)^r\left[\frac{(n+r)!}{n!}f_n(t)\right.\\ &-r\frac{(n+r)!}{(n-1)!\,(n+1)}f_{n+1}(t)+\frac{r(r-1)}{2!}\frac{(n+r)!}{(n-1)!\,(n+2)}f_{n+2}(t)\\ &\left.-\cdots(-1)^r\frac{(n+r)!}{(n-1)!\,(n+r)}f_{n+r}(t)\right],\end{aligned} \tag{5}$$

where the c's are determined by (3) as explained.

The first $r+1$ terms are independent of n and bounded for all $t > 0$. Therefore, in order to show condition 1′ of Lemma 1 is satisfied, we have only to consider

$$\begin{aligned}&\frac{(n+r)!}{n!}f_n(t)-r\frac{(n+r)!}{(n-1)!\,(n+1)}f_{n+1}(t)\\ &+\frac{r(r-1)}{2!}\frac{(n+r)!}{(n-1)!\,(n+2)}f_{n+2}(t)-\cdots(-1)^r\frac{(n+r)!}{(n-1)!\,(n+r)}f_{n+r}(t).\end{aligned} \tag{6}$$

We can write this in the form

$$\frac{(n+r)!}{(n-1)!}\left[\frac{f_n(t)}{n} - r\frac{f_{n+1}(t)}{n+1} + \frac{r(r-1)}{2!}\frac{f_{n+2}(t)}{n+2} - \cdots (-1)^r \frac{f_{n+r}(t)}{(n+r)}\right].$$

We denote $\frac{f_n(t)}{n}$ by $\varphi(n, t)$ and have

$$\frac{(n+r)!}{(n-1)!}\left[\varphi(n, t) - r\varphi(n+1, t) + \frac{r(r-1)}{2!}\varphi(n+2, t)\right.$$
$$\left. - \cdots (-1)^r\varphi(n+r, t)\right] = \pm\frac{(n+r)!}{(n-1)!}[\varphi^{(r)}(n+\theta_r, t)],$$

where $0 < \theta_r < r$, by Lemma 3. We rewrite this as*

$$\frac{(n+r)!}{(n-1)!\,(n+\theta_r)^{r+1}}[(n+\theta_r)^{r+1}\varphi^{(r)}(n+\theta_r, t)].$$

The first factor is bounded for all n and all $t > 0$ as

$$\lim_{n\to\infty}\frac{(n+r)!}{(n-1)!\,(n+\theta_r)^{r+1}} = 1,$$

uniformly in t, since $0 < \theta_r < r$. Therefore, we need only consider the other factor. It can be written*

$$x^{r+1}\varphi^{(r)}(x, t), \quad \text{where} \quad \varphi(x, t) = \frac{f(x, t)}{x} \quad \text{and} \quad f(x, t) = e^{-tx\log x}.$$

From Lemma 2, we have

$$x^{r+1}\varphi^{(r)}(x, t) = \sum_{k=0}^{r}(-1)^k\frac{r!}{(r-k)!}x^{r-k}f^{(r-k)}(x, t)$$
$$= x^r f^{(r)}(x, t) - rx^{r-1}f^{(r-1)}(x, t) + \cdots (-1)^{r-1}r!\,f(x, t),$$

where $f^{(r)}(x, t)$ is given in sufficient detail by Lemma 4. We can surely write

$$(7) \quad |x^{r+1}\varphi^{(r)}(x, t)| \leq x^r|f^{(r)}(x, t)| + rx^{r-1}|f^{(r-1)}(x, t)| + \cdots + r!\,|f(x, t)|.$$

Call $tx\log x = \beta$. Then $tx(1+\log x) \leq 2tx\log x = 2\beta$. The first term of (7) gives, by Lemma 4,

$$|x^r f^{(r)}(x, t)| = e^{-\beta}\left[O(\beta^r) + \frac{1}{\log x}O(\beta^{r-1}) + \frac{1}{\log x}O(\beta^{r-2})\right.$$
$$\left. + \cdots + \frac{1}{\log x}O(\beta)\right] = e^{-\beta}\left[O(\beta^r) + \frac{1}{\log x}O(\beta^{r-1})\right].$$

For constant p, $\frac{\beta^p}{e^\beta}$ is bounded for all β (since it has a maximum for $\beta = p$

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* The notation $\varphi^{(r)}(x, t)$ denotes differentiation with respect to the first argument.

and $\to 0$ as $\beta \to \infty$). It follows that

$$|x^r f^{(r)}(x, t)| \leq A + \frac{B}{\log x} < C,$$

where A, B, C are independent of x and t.

The other terms of (7), except the last, are exactly of the nature of the term investigated and consequently each of them is less than some constant, independent of x and t. The last term is $r!\,f(x, t) = r!\,e^{-tx \log x}$ which $< r!$ for all positive t and large positive x. Therefore, since the number of terms in (7) is finite, we can say $|x^{r+1}\varphi^{(r)}(x, t)| \leq M$, where M is independent of x and t. Therefore, expression (6) is bounded for all n and all $t > 0$, and condition 1 of Lemma 1 is satisfied.

It remains to show that condition 2 of Lemma 1 is satisfied. We desire to show that, for each constant $t > 0$, $\dfrac{(n + r)!}{n!}\Delta^{r+1}e^{-nt \log n}$ changes sign but a finite number of times and that this number is bounded independently of t. Since n is a positive integer, we need only consider

$$\Delta^{r+1}e^{-nt \log n}.$$

Lemma 3 states that

$$\Delta^r\varphi(n) = \varphi^{(r)}(n + \theta_r), \qquad 0 < \theta_r < r.$$

We therefore consider the $(r + 1)$th derivative of $e^{-tx \log x}$.

Writing $\xi = 1 + \log x$,

$$f^{(r)}(x, t) = e^{-tx \log x}Q_r.$$

Since

$$f^{(r+1)}(x, t) = e^{-tx \log x}[-t\xi Q_r + Q_r'],$$

$$\text{(A)} \qquad Q_{r+1} = Q_r' - t\xi Q_r.$$

We now say that

$$\text{(B)} \qquad \begin{aligned} Q_r = &- \xi^{(r-1)}t + [r\xi\xi^{(r-2)} + \psi_{r-2}]t^2 \\ &- \left[\frac{r(r-1)}{2!}\xi^2\xi^{(r-3)} + \psi_{r-3}\right]t^3 + \cdots \\ &+ (-1)^{r-1}\left[\frac{r!}{(r-2)!\,2!}\xi^{r-2}\xi'\right]t^{r-1} + (-1)^r\xi^r t^r; \end{aligned}$$

where ψ_p is a polynomial in $\xi, \xi', \cdots, \xi^{(p-1)}$, in which the sum of the orders of the derivatives in each term is p, repetitions being counted, and in which the exponent of the highest power of ξ is less than $r - p - 1$, the exponent of the power of ξ in the first term of the same coefficient. This can be proved by mathematical induction, using (A).

We have, by differentiation,

$$Q_r' = -\xi^{(r)}t + [r\xi\xi^{(r-1)} + \psi_{r-1}]t^2 - \cdots + (-1)^p\left[\frac{r!}{(p-1)!\,(r-p+1)!}\xi^{p-1}\xi^{(r-p+1)} + \psi_{r-p+1}\right]t^p + \cdots + (-1)^r r\xi^{r-1}\xi' t^r.$$

We consider the expression for the coefficient of t^p in Q_r'. We recall that $\xi = 1 + \log x$, so that $\xi^{(p)} = (-1)^{p-1}\dfrac{(p-1)!}{x^p}$. The coefficient of t^p has a factor $\dfrac{1}{x^{r-p+1}}$. This is a factor of ψ_{r-p+1} because the sum of the orders of the derivatives for each term of polynomial ψ_{r-p+1} is $r - p + 1$. If we remove this factor, we have

$$\text{(C)} \qquad \frac{1}{x^{r-p+1}}\left[(-1)^r\frac{r!}{(r-p+1)!\,(p-1)!}\xi^{p-1} + P_{p-2}(\xi)\right]$$

where $P_{p-2}(\xi)$ is a polynomial in ξ of degree $\leq p - 2$. This is true because ψ_{r-p+1} contains ξ to a power not higher than $p - 2$.

ξ becomes positively infinite as $x \to \infty$. Therefore, we can select a value for x, say N_p, such that, for $x > N_p$, the sign of the expression (C) will be the sign of the first term; namely $(-1)^r$, as x, r, p, ξ are positive, $x > 1$. This sign is independent of p; thus, if for each p we select N_p so that the first term in each coefficient will dominate and then choose the largest of the set of values N_p, say N, we can say that for $x > N$ each coefficient of Q_r' will have the sign $(-1)^r$. It follows then that for $x > N$ and for all values of $t > 0$, Q_r' has one sign. This N is then independent of t. Therefore, Q_r can change sign not more than once, $x > N$. It follows then that $f^{(r)}(x, t) = f(x, t)Q_r$ changes sign not more than once, $x > N$. The location of this change of sign, if any, will generally depend on t, but N is independent of t.

Since this is true for any value of r, we have that $f^{(r+1)}(x, t)$ changes sign not more than once, $x > N$. We can then say that $\Delta^{r+1}e^{-nt\log n}$ can change sign not more than once for $n > N'$; where N' is the value of n corresponding to the value N for the $(r + 1)$th derivative of $e^{-nt\log n}$. This N' can be taken independent of t also, since the two values differ by θ_r, where $0 < \theta_r < r$. Since we consider only integer values of n and since there are only a finite number of integers less than N', it follows that $\Delta^{r+1}e^{-nt\log n}$ can change sign but a finite number of times, bounded in t, for $t > 0$. Thus condition 2 of Lemma 1 is satisfied and the theorem is proved.

We can now state

THEOREM II. *The Dirichlet's series definition $D_{n \log n}$ includes C_r for all r, real or complex, $R(r) > -1$.*

It is known that if a series is summable C_r, it is summable C_s provided $R(s) > R(r) > -1$.* Theorem II follows immediately from this. When $R(r) \leq 0$, the regularity of the definition is sufficient to insure that it includes C_r.

3. THE TRANSFORMATION $D_{n \log n}$ FROM A DIRICHLET'S SERIES VIEWPOINT.

A Dirichlet's series† is a series of the form

$$f(t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t},$$

where λ_n is a sequence of real increasing numbers whose limit is infinite; and in the cases we consider, $\lambda_1 \geq 0$. When t is complex, we write

$$t = \sigma + i\tau.$$

If we apply the transformation $D_{n \log n}$ to a series $\sum_{n=1}^{\infty} u_n$, we get

$$y(t) = \sum_{n=1}^{\infty} u_n e^{-nt \log n}$$

which is a Dirichlet's series, $\lambda_n = n \log n$. We now make use of this property of the transformation $D_{n \log n}$ to prove a theorem on regularity and to prove, in a much shorter way than in section 2, that $D_{n \log n}$ includes C_r.

THEOREM I. *The Dirichlet's series definition, $D_{n \log n}$, is regular if $t \to 0$ over a point set, T, which lies entirely within an angle whose vertex is at the origin such that $|\text{am}\, t| \leq a < \pi/2$.*

We assume $\sum_{n=1}^{\infty} u_n$ convergent and prove that $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges and $\lim\limits_{\substack{t \to 0 \\ T}} \sum_{n=1}^{\infty} u_n e^{-nt \log n} = \sum_{n=1}^{\infty} u_n$.

Since $\sum_{n=1}^{\infty} u_n$ converges, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges for $t = 0$ and to the value $\sum_{n=1}^{\infty} u_n$. Hardy and Riesz‡ prove that if a Dirichlet's series is convergent for $t = t_0$, then it is uniformly convergent throughout the angular

* Chapman, *Proceedings of the London Mathematical Society*, Series 2, Vol. 9 (1910–11), p. 369. See also *Report*, p. 27.

† Hardy and Riesz, *The General Theory of Dirichlet's Series* (Cambridge Tracts in Mathematics and Mathematical Physics, No. 18).

‡ Loc. cit., Theorem 2.

region in the plane of t defined by the inequality

$$|\text{am }(t - t_0)| \leq \text{a} < \pi/2.$$

Therefore, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges uniformly in this region, say to the value $f(t)$. It follows from this uniform convergence that if t approach 0 over any point set T in the region defined, $f(t)$ will approach $f(0)$ which is $\sum_{n=1}^{\infty} u_n$ and the theorem is proved.

Riesz* has given the following definition for the sum of a divergent series, $\sum_{n=1}^{\infty} u_n$. Form

$$y(t) = t^{-k} \sum_{\lambda_n < t} (t - \lambda_n)^k u_n$$

in which λ_n is a sequence of real increasing numbers whose limit is infinite, $\lambda_1 \geq 0$. Then the generalized value of the series is given by

$$\lim_{\substack{t \to \infty \\ T}} y(t)$$

where T is a set of positive real numbers. If this limit exists, the series is said to be summable (λ, k). This definition is regular, $k \geq 0$.

Riesz has shown† that, if $\lambda_n = n$, then this definition is equivalent to Cesàro's means and to the generalizations of the Cesàro means due to Knopp and Chapman.

LEMMA: *If* $\sum_{n=1}^{\infty} u_n$ *is summable* (n, k), *it is summable* $(n \log n, k)$ *to the same value.*

This can be shown by use of a theorem due to Hardy:‡

If the series $\sum_{n=1}^{\infty} u_n$ is summable (λ, k) to the sum l, and if μ is a logarithmico-exponential function of λ such that $\mu = O(\lambda^{\Delta})$, where Δ is a constant; then the series, $\sum_{n=1}^{\infty} u_n$, is summable (μ, k) to the sum l.

In this case, let $\mu = n \log n$ and $\lambda = n$; then the condition $n \log n = O(n^{\Delta})$ is true for any constant $\Delta > 1$. The hypotheses of the theorem are satisfied and the lemma is proved.

We are now in a position to prove

THEOREM II. *The Dirichlet's series definition,* $D_{n \log n}$, *when* t *approaches* 0 *over a point set* T, *lying within an angle with vertex at the origin such that* $|\text{am } t| \leq \text{a} < \pi/2$, *includes the Cesàro means of all orders.*

* *Comptes Rendus*, Vol. 149 (1909), p. 910.

† *Comptes Rendus*, Vol. 152 (1911), p. 1651.

‡ *Proceedings of London Mathematical Society*, Series 2, Vol. 15 (1916), p. 72.

To prove this theorem, it is necessary to show that if $\sum_{n=1}^{\infty} u_n$ is summable C_k to the sum l, then $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ converges, $R(t) > 0$, and $\lim_{t \to 0} \sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is l.

We assume that $\sum_{n=1}^{\infty} u_n$ is summable C_k and, consequently, summable (n, k) to the sum l, where k is the order of summability. It is evident that, for $t = 0$, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is summable (n, k) and, by the lemma, $(n \log n, k)$ and that the sum is l. It is also summable $(n \log n, k)$ for all t whose real part is greater than 0, say to the sum $f(t)$. This is due to a theorem by Hardy and Riesz* which states that if a Dirichlet's series, $\sum_{n=1}^{\infty} u_n e^{-\lambda_n t}$, is summable (λ, k) for a value of t whose real part is σ, then it is summable (λ, k) for all values of t whose real part is greater than σ.

Hardy and Riesz† also proved that if a series is summable (λ, k) for $t = t_0$, and has sum $f(t_0)$, and if the series has sum $f(t)$ when the real part of t is greater than the real part of t_0, then $f(t) \to f(t_0)$ as $t \to t_0$ along any path lying entirely within the angle whose vertex is at t_0 and such that $|\operatorname{am}(t - t_0)| \leq a < \pi/2$. We have then that, if t approaches zero along any path in the angle defined, $f(t) \to l$.

But, we shall show that $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is actually convergent, $R(t) > 0$. Since $\sum_{n=1}^{\infty} u^n$ is summable C_k, $u_n/n^k \to 0$ as $n \to \infty$. The series $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ has a region of absolute convergence coinciding with its region of convergence.‡ Therefore, we can write

$$|u_n e^{-nt \log n}|^{1/n} = \left|\frac{u_n}{n^k}\right|^{1/n} \left|\frac{n^{k/n}}{n^t}\right| = \left|\frac{u_n}{n^k}\right|^{1/n} |n^{(k/n)-t}| \to 0 \quad \text{for} \quad R(t) > 0.$$

Therefore, by Cauchy's test $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ is convergent, $R(t) > 0$.

Because of the regularity of $(n \log n, k)$, $\sum_{n=1}^{\infty} u_n e^{-nt \log n}$ must be summable $(n \log n, k)$ to the same $f(t)$ to which it converges. It follows then that the value approached as $t \to 0$ must be the same in both cases. There-

* Loc. cit., Theorem 25.

† Loc. cit., Theorem 28.

‡ Hardy and Riesz, loc. cit., Theorem 9.

fore,

$$\lim_{t\to 0}\sum_{n=1}^{\infty} u_n e^{-nt\log n} = l$$

and the theorem is proved.

4. SOME RELATIONS BETWEEN THE BOREL DEFINITION, THE LEROY DEFINITION, AND THE DIRICHLET'S SERIES DEFINITION $D_{n\log n}$.

THEOREM I. *LeRoy's definition does not include Borel's definition.*

We prove this theorem by giving a series that is summable Borel and not summable LeRoy. Let us consider the series

$$\sum_{n=0}^{\infty} u_n = 1 + 0 - 2 + 0 + 3\cdot 4 + 0 - 4\cdot 5\cdot 6 + 0 + \cdots,$$

where

$$\begin{cases} u_n = 0, \quad n \text{ odd}, \\ u_n = (-1)^{n/2}\dfrac{n!}{\left(\dfrac{n}{2}\right)!}, \quad n \text{ even}, \end{cases}$$

This series is summable Borel, for

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{u_n t^n}{n!} &= 1 + 0 - \frac{2t^2}{2!} + 0 + \frac{3\cdot 4t^4}{4!} + 0 \cdots \\ &= 1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \cdots \\ &= e^{-t^2},\end{aligned}$$

and

$$\int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}\frac{u_n t^n}{n!}\,dt = \int_0^{\infty} e^{-t^2-t}dt$$

which converges, since $e^{-t^2-t} \leq e^{-t}$, both are positive, and $\int_0^{\infty} e^{-t}dt$ converges. Therefore, the series is summable Borel.

This series is not summable LeRoy. If we apply the LeRoy transformation to this series, we get

$$L = 1 + 0 - \frac{\Gamma(2t+1)}{2!}2 + 0 + \frac{\Gamma(4t+1)}{4!}3\cdot 4 + 0 - \frac{\Gamma(6t+1)}{6!}4\cdot 5\cdot 6 + 0 + \frac{\Gamma(8t+1)}{8!}5\cdot 6\cdot 7\cdot 8 \cdots$$

so that, if V_n is the nth term of this series,

$$\begin{cases} V_n = 0, \quad n \text{ odd}, \\ V_n = (-1)^{n/2}\dfrac{\Gamma(nt+1)}{\left(\dfrac{n}{2}\right)!}, \quad n \text{ even}. \end{cases}$$

In order to see what happens to the nth term of this series, as $n \to \infty$, we write $n = 2k$, so that

$$V_{2k} = (-1)^k \frac{\Gamma(2kt+1)}{k!}$$

and find the limit approached by V_{2k} as $k \to \infty$. By Stirling's formula

$$\frac{\Gamma(2kt+1)}{k!} = 2^{2kt+(1/2)} k^{k(2t-1)} t^{2kt+(1/2)} e^{-k(2t-1)}[1+o(1)],$$

so that

$$\left[\frac{\Gamma(2kt+1)}{k!}\right]^{1/2k} = 2^{t+(1/4k)} k^{t-(1/2)} t^{t+(1/4k)} e^{-t+(1/2)}[1+o(1)].$$

For a fixed $t > \frac{1}{2}$,

$$\left[\frac{\Gamma(2kt+1)}{k!}\right]^{1/2k} \to \infty \quad \text{as } k \to \infty.$$

It follows then that the series L cannot converge, $t > \frac{1}{2}$, since lim sup $|V_n| = \infty$. Since the series L does not converge, $t_0 < t < 1$, it is not summable LeRoy and the theorem is proved.

Before proceeding to the next theorem, we prove two lemmas.

LEMMA 1. *If* $\sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ *converges, then* $\sum_{n=0}^{\infty} \frac{\Gamma(nt'+1)}{\Gamma(n+1)} u_n$ *converges absolutely,* $t' < t$.

We assume that $\sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ is convergent. Therefore,

$$\frac{\Gamma(nt+1)}{\Gamma(n+1)} |u_n| < C, \quad |u_n| < C \frac{\Gamma(n+1)}{\Gamma(nt+1)},$$

and

$$\frac{\Gamma(nt'+1)}{\Gamma(n+1)} |u_n| \le \frac{\Gamma(nt'+1)}{\Gamma(n+1)} \cdot C \frac{\Gamma(n+1)}{\Gamma(nt+1)} = \frac{\Gamma(nt'+1)}{\Gamma(nt+1)} C$$
$$= n^{n(t'-t)} e^{-n(t'-t)} \frac{t'^{nt'+(1/2)}}{t^{nt+(1/2)}} C[1+o(1)],$$

then

$$\left|\frac{\Gamma(nt'+1)}{\Gamma(n+1)} u_n\right|^{1/n} \le n^{t'-t} e^{-(t'-t)} \frac{t'^{t'+(1/2n)}}{t^{t+(1/2n)}} C^{1/n}[1+o(1)]$$
$$< kn^{t'-t}[1+o(1)] \to 0, \text{ as } n \to \infty,\ t' < t.$$

Thus Lemma 1 is proved.

LEMMA 2. *Let* $k(a, x)$ *be continuous,* $a > 0$, $x > 0$, *and let* $U(x)$ *be a continuous function such that* $\lim_{x \to \infty} U(x) = l$. *If*

1. $\int_0^{\infty} |k(a, x)| dx$ *exists for each* $a > 0$ *and is bounded for all* $a > 0$,

2. $\lim\limits_{a \to 0} \int_{\alpha}^{\beta} |k(a, x)| dx = 0, \ \beta > \alpha > 0,$

3. $\lim\limits_{a \to 0} \int_{0}^{\alpha} k(a, x) dx = -1, \ \alpha > 0,$

4. $\lim\limits_{a \to 0} \int_{0}^{\alpha} k(a, x) dx = 0,$

then $\int_0^\infty k(a, x) U(x) dx$ *exists for each* $a > 0$ *and*

$$\lim_{a \to 0} \int_0^\infty k(a, x) U(x) dx = l - U(0).$$

The fact that $\int_0^\infty k(a, x) U(x) dx$ exists follows immediately from condition 1 and the fact that $U(x)$ is bounded.

We can write

$$\int_0^\infty k(a, x) U(x) dx = \int_0^\eta k(a, x)[U(x) - U(0)] dx + U(0) \int_0^\eta k(a, x) dx + \int_\eta^\xi k(a, x) U(x) dx + \int_\xi^\infty k(a, x)[U(x) - l] dx + \int_\xi^\infty k(a, x) dx,$$

where ξ, η are constants, each independent of a, such that

$$|U(x) - l| < \epsilon, \quad x > \xi$$
$$|U(x) - U(0)| < \epsilon, \quad x < \eta.$$

Because of (3)

$$U(0) \int_0^\eta k(a, x) dx = -U(0) + h_1(a), \qquad \text{where} \qquad \lim_{a \to 0} h_1(a) = 0,$$

and because of (3) and (4)

$$l \int_\xi^\infty k(a, x) dx = l + h_2(a), \qquad \text{where} \qquad \lim_{a \to 0} h_2(a) = 0.$$

It follows that

$$\left| \int_0^\infty k(a, x) U(x) dx + U(0) - l \right| \leq \int_0^\eta |k(a, x)| \, |U(x) - U(0)| dx + \int_\eta^\xi |k(a, x)| \, |U(x)| dx + \int_\xi^\infty |k(a, x)| \, |U(x) - l| dx.$$

By (1) and the choice of η

$$\int_0^\eta |k(a, x)| \, |U(x) - U(0)| dx \leq \epsilon \cdot C;$$

by (2) and the fact that $|U(x)| < M$

$$\int_\eta^\xi |k(a, x)|\,|U(x)|dx \leq M\cdot h_3(a), \quad \text{where} \quad \lim_{a\to 0} h_3(a) = 0;$$

by (1) and the choice of ξ

$$\int_\xi^\infty |k(a, x)|\,|U(x) - l|dx \leq \epsilon\cdot A.$$

It follows that

$$\left|\int_0^\infty k(a, x)U(x)dx + U(0) - l\right| \leq \epsilon\cdot C + M\cdot h_3(a) + \epsilon\cdot A + |h_1(a)| + |h_2(a)|,$$

so that

$$\limsup_{a\to 0}\left|\int_0^\infty k(a, x)U(x)dx + U(0) - l\right| \leq \epsilon\cdot C + \epsilon\cdot A.$$

Since this limit is less than an arbitrarily chosen small positive constant, it follows that

$$\lim_{a\to 0}\int_0^\infty k(a, x)U(x)dx = l - U(0).$$

THEOREM II. *If a series,* $\sum_{n=0}^\infty u_n$, *is summable Borel and if* $\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ *converges,* $t_0 < t < 1$, *then the series is summable LeRoy, and to the same value.*

Bromwich proved* this theorem under the assumption that

$$\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$$

converges absolutely. Because of Lemma 1, we need not require the absolute convergence of this series but simply its convergence. The following proof differs materially from the one given by Bromwich.

We assume that $\int_0^\infty e^{-x}u(x)dx$ converges to the value l, and desire to prove that, if $\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ converges, then $\lim_{t\to 1^-}\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n = l$. We can express $\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ in terms of an integral as follows:

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, \qquad \frac{\Gamma(nt+1)}{n!} = \int_0^\infty e^{-x}\frac{x^{nt}}{n!}dx,$$

and

$$\sum_{n=0}^\infty \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n = \sum_{n=0}^\infty \int_0^\infty e^{-x}\frac{u_n x^{nt}}{n!}dx = \int_0^\infty e^{-x}\sum_{n=0}^\infty \frac{u_n x^{nt}}{n!}dx,$$

* Bromwich, *Infinite Series* (London, 1908).

where the term-by-term integration is justified by Bromwich* under the hypothesis of absolute convergence which is equivalent, because of Lemma 1, to ours.

We can now write

$$\lim_{t\to 1^-}\sum_{n=0}^{\infty}\frac{\Gamma(nt+1)}{\Gamma(n+1)}u_n = \lim_{t\to 1^-}\int_0^\infty e^{-x}u(x^t)dx, \quad \text{where} \quad u(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}u_n.$$

Now, if the change of variable $x = y^{1/t}$ is made, and if we then replace y by x and $1/t - 1$ by a, we have

$$\lim_{t\to 1^-}\int_0^\infty e^{-x}u(x^t)dx = \lim_{a\to 0^+}\int_0^\infty e^{-x^{1+a}}x^a u(x)dx + \lim_{a\to 0^+} a\int_0^\infty e^{-x^{1+a}}x^a u(x)dx.$$

It now remains to show that if $\int_0^\infty e^{-x}u(x)dx$ converges to the value l, then $\int_0^\infty e^{-x^{1+a}}x^a u(x)dx$ converges for all values of a in an interval $(0, a)$ and that $\lim_{a\to 0}\int_0^\infty e^{-x^{1+a}}x^a u(x)dx = l$. If this is proved, it will be clear that the second integral in the above equation $\to 0$ as $a \to 0$.

Let us write $U(x) = \int_0^x e^{-s}u(s)ds$. Then it follows that $\lim_{x\to\infty} U(x) = l$ and that $U'(x) = e^{-x}u(x)$. Therefore,

$$\int_0^\infty e^{-x^{1+a}}x^a u(x)dx = \int_0^\infty e^{x-x^{1+a}}x^a U'(x)dx.$$

Integration by parts gives

$$\int_0^\infty e^{x-x^{1+a}}x^a U'(x)dx = \lim_{\xi\to\infty}\left[e^{x-x^{1+a}}x^a U(x)\right]_0^\xi - \lim_{\xi\to\infty}\int_0^\xi \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx.$$

The last integral converges as $\xi \to \infty$ and we have

$$\int_0^\infty e^{-x^{1+a}}x^a u(x)dx = -\int_0^\infty \frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx.$$

Lemma 2 can now be used to show that whenever $\lim_{x\to\infty} U(x) = l$, then

$$\int_0^\infty -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx$$

exists for each $a > 0$ and

$$\lim_{a\to 0}\int_0^\infty -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]U(x)dx = l.$$

Here

$$k(a, x) = -\frac{d}{dx}\left[e^{x-x^{1+a}}x^a\right]$$

is continuous, $a > 0$, $x > 0$. $U(x)$ is an indefinite integral and therefore continuous, $0 < x$, and $\lim_{x\to\infty} U(x) = l$. It remains to show that $k(a, x)$ satisfies the four conditions of Lemma 2.

In order to show that condition 1 is satisfied, it is desirable to know where $k(a, x)$ changes sign. We find

$$k(a, x) = -\frac{d}{dx}[e^{x-x^{1+a}}x^a] = -e^{x-x^{1+a}}x^{a-1}[x + a - (a + 1)x^{a+1}].$$

Since the factor outside the bracket is negative, we investigate $x + a - (1 + a)x^{1+a}$. It vanishes when $x = 1$; and its second derivative < 0, $x > 0$. Since the function > 0 when $x = 0$, it follows that it changes sign only when $x = 1$. Therefore, $k(a, x)$ is negative, $0 < x < 1$, and positive, $x > 1$. We now show that the four conditions of Lemma 2 are satisfied.

Condition 1.

$$\int_0^\infty \left| -\frac{d}{dx}[e^{x-x^{1+a}}x^a] \right| dx = \int_0^1 \frac{d}{dx}[e^{x-x^{1+a}}x^a]dx - \int_1^\infty \frac{d}{dx}[e^{x-x^{1+a}}x^a]dx$$
$$= [e^{x-x^{1+a}}x^a]_0^1 - [e^{x-x^{1+a}}x^a]_1^\infty = 2.$$

Condition 2.

If $\alpha < \beta \leq 1$ or $1 \leq \alpha < \beta$, then

$$\lim_{a\to 0} \int_\alpha^\beta \left| -\frac{d}{dx}[e^{x-x^{1+a}}x^a] \right| dx = \lim_{a\to 0} \int_\alpha^\beta \pm \frac{d}{dx}[e^{x-x^{1+a}}x^a]dx$$
$$= \pm \lim_{a\to 0} [e^{\beta-\beta^{1+a}}\beta^a - e^{\alpha-\alpha^{1+a}}\alpha^a] = 0.$$

If $\alpha < 1 < \beta$, then

$$\lim_{a\to 0} \int_\alpha^\beta \left| -\frac{d}{dx}[e^{x-x^{1+a}}x^a] \right| dx = \lim_{a\to 0} \int_\alpha^1 \frac{d}{dx}[e^{x-x^{1+a}}x^a]dx$$
$$- \lim_{a\to 0} \int_1^\beta \frac{d}{dx}[e^{x-x^{1+a}}x^a]dx = \lim_{a\to 0}[e^{x-x^{1+a}}x^a]_\alpha^1 - \lim_{a\to 0}[e^{x-x^{1+a}}x^a]_1^\beta = 0.$$

Condition 3.

If $\alpha > 0$,

$$\lim_{a\to 0} \int_0^\alpha -\frac{d}{dx}[e^{x-x^{1+a}}x^a]dx = \lim_{a\to 0} [-e^{x-x^{1+a}}x^a]_0^\alpha = -1.$$

Condition 4.

$$\int_0^\infty -\frac{d}{dx}[e^{x-x^{1+a}}x^a]dx = [-e^{x-x^{1+a}}x^a]_0^\infty = 0.$$

The conditions of the lemma are then satisfied and we can say that if $\int_0^\infty e^{-x}u(x)dx$ converges to the value l, then

$$\int_0^\infty e^{-x^{1+a}}x^a u(x)dx$$

converges, $a > 0$, and

$$\lim_{a \to 0} \int_0^\infty e^{-x^{1+\epsilon}} x^a u(x) dx = l - U(0).$$

But $U(0) = 0$, and the theorem is proved.

COROLLARY. *The LeRoy and Borel definitions are mutually consistent.**

THEOREM III. *A sufficient condition that* $\sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n$ *shall converge,* $t_0 < t < 1$, is that $\lim_{n \to \infty} \frac{|u_n|^{1/n}}{n^\epsilon} = 0, \epsilon > 0$.

By Stirling's formula

$$\frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n = n^{n(t-1)} t^{nt+(1/2)} e^{-n(t-1)} [1 + o(1)] u_n.$$

Therefore,

$$\left| \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n \right|^{1/n} = n^{t-1} t^{t+(1/2n)} e^{-(t-1)} [1 + o(1)] |u_n|^{1/n} \to 0$$

under our hypothesis, $t_0 < t < 1$.

THEOREM IV. *If a given series,* $\sum_{n=0}^{\infty} u_n$, *is summable Borel and if* $\lim_{n \to \infty} \frac{|u_n|^{1/n}}{n^\epsilon} = 0, \epsilon > 0$, *then the series is summable LeRoy to the same value.*

The proof follows immediately from Theorem II and Theorem III.

THEOREM V. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* C_r *and summable Borel,† it is summable LeRoy to the same value.*

Since $\sum_{n=0}^{\infty} u_n$ is summable C_r, it is necessary that $\frac{u_n}{n^r} \to 0$ as $n \to \infty$. Therefore,

$$|u_n| < Cn^r, \qquad |u_n|^{1/n} < C^{1/n} n^{r/n}, \qquad \text{and}$$

$$\frac{|u_n|^{1/n}}{n^\epsilon} \leq \frac{C^{1/n} n^{r/n}}{n^\epsilon} \leq C^{1/n} n^{(r/n)-\epsilon} \to 0, \quad n \to \infty;$$

and, by Theorem IV, the theorem is proved.

THEOREM VI. *If* $\sum_{n=0}^{\infty} u_n$ *is summable* E_r§ *and Borel,‡ it is summable LeRoy to the same value.*

Since $\sum_{n=0}^{\infty} u_n$ is summable E_r, it is necessary that $\frac{u_n}{(2r-1)^n} \to 0$ as $n \to \infty$.*

* Ricotti has proved this corollary in another way. See *Giornale di Matematiche*, Vol. 48 (1910), p. 80.

† Necessarily to the same value because of mutual consistency.

‡ *Report*, p. 32.

§ Necessarily to the same value because of mutual consistency.

Therefore,

$$|u_n|^{1/n} < C^{1/n}(2r-1),$$

and

$$\frac{|u_n|^{1/n}}{n^\epsilon} \leq \frac{C^{1/n}(2r-1)}{n^\epsilon} \to 0, \quad \text{as} \quad n \to \infty,$$

and, by Theorem IV, the theorem is proved.

THEOREM VII. *If $\sum_{n=0}^{\infty} u_n$ is summable E_r, it is summable LeRoy to the same value.*

This theorem states that LeRoy's definition includes the definition given by the exponential mean, E_r. Borel's definition includes E_r.* Therefore, every series summable E_r is summable Borel and consequently, by Theorem VI, is summable LeRoy.

THEOREM VIII. *If $\sum_{n=0}^{\infty} u_n$ is summable by the Euler power series definition*† *and summable Borel,*‡ *it is summable LeRoy to the same value.*

Since $\sum_{n=0}^{\infty} u_n$ is summable by the Euler power series definition, it is necessary that $\limsup_{n\to\infty} |u_n|^{1/n} \leq 1.$§ Therefore,

$$\lim_{n\to\infty} \frac{|u_n|^{1/n}}{n^\epsilon} \leq \lim_{n\to\infty} \frac{1}{n^\epsilon} = 0, \quad \epsilon > 0,$$

and the theorem is proved.

THEOREM IX. *If $\sum_{n=0}^{\infty} u_n$ is summable $D_{n \log n}$ and Borel, then it is summable LeRoy to the value to which it is summable Borel.*

Given an $\epsilon > 0$, then it is necessary if the $D_{n \log n}$ series converges, $t > 0$, that

$$|u_n| e^{-nt \log n} < A, \quad t > 0,$$

and, in particular,

$$|u_n| e^{-n(\epsilon/2) \log n} < A.$$

Then

$$|u_n| < A e^{n(\epsilon/2) \log n}$$

and

$$|u_n|^{1/n} < A^{1/n} e^{\epsilon/2 \log n} = A^{1/n} n^{\epsilon/2}.$$

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* *Report*, p. 27.

† *Report*, p. 23.

‡ Necessarily to the same value because of mutual consistency.

§ *Report*, p. 32.

Therefore,

$$\frac{|u_n|^{1/n}}{n^{\epsilon}} < \frac{A^{1/n} n^{\epsilon/2}}{n^{\epsilon}} = \frac{A^{1/n}}{n^{\epsilon/2}} \to 0,$$

and, by Theorem IV, the theorem is proved.

THEOREM X. *The Dirichlet's series definition* $D_{n \log n}$ *does not include Borel's definition.*

This is shown by means of the series

$$\sum_{n=1}^{\infty} u_n = 1 + 0 - 2 + 0 + 3 \cdot 4 + 0 - 4 \cdot 5 \cdot 6 + \cdots,$$

where

$$\begin{cases} u_n = 0, & n \text{ even}, \\ u_n = (-1)^{(n-1)/2} \dfrac{(n-1)!}{\left(\dfrac{n-1}{2}\right)!}, & n \text{ odd}. \end{cases}$$

It was shown in the proof of Theorem I of this section that this series is Borel summable. It is not summable $D_{n \log n}$. To show this, we form

$$\sum_{n=1}^{\infty} u_n e^{-nt \log n}$$

and show that it is not convergent $t > 0$. The nth term is

$$\begin{cases} 0, & n \text{ even}, \\ (-1)^{(n-1)/2} \dfrac{(n-1)!}{\left(\dfrac{n-1}{2}\right)!} e^{-nt \log n}; & n \text{ odd}. \end{cases}$$

We take the superior limit of the nth root of the nth term as $n \to \infty$.

$$\limsup_{n \to \infty} \left| \frac{(n-1)!}{\left(\dfrac{n-1}{2}\right)!} e^{-nt \log n} \right|^{1/n} = \begin{cases} 0, & t \geq \frac{1}{2}, \\ \infty, & t < \frac{1}{2}. \end{cases}$$

By Stirling's formula

$$(n-1)! = \Gamma(n) = \sqrt{2\pi}\, n^{n-(1/2)} e^{-n}[1 + o(1)],$$

and

$$\frac{n-1}{2}! = \Gamma\left(\frac{n+1}{2}\right) = \sqrt{2\pi} \left(\frac{n+1}{2}\right)^{(n+1)/2-1/2} e^{-(n+1)/2}[1 + o(1)],$$

and

$$\left| \frac{(n-1)!}{\left(\dfrac{n-1}{2}\right)!} e^{-nt \log n} \right|^{1/n} = \sqrt{2}\, n^{1/2-1/2n} \left(1 + \frac{1}{n}\right)^{-1/2} e^{-1/2+1/2n} n^{-t}[1 + o(1)].$$

$$= \sqrt{2}\, n^{1/2-t-1/2n}\left(1+\frac{1}{n}\right)^{-1/2} e^{-1/2+1/2n}[1+o(1)]$$

$$= \begin{cases} 0, & t \geq \frac{1}{2} \\ \infty, & t < \frac{1}{2}. \end{cases}$$

Since for this Dirichlet's series the region of convergence coincides with the region of absolute convergence,* it follows that this series does not converge, $0 < t < \frac{1}{2}$, and the theorem is proved.

THEOREM XI. *Borel's definition does not include $D_{n \log n}$.*

We have proved that $D_{n \log n}$ includes C_r. A series summable C_r and not Borel would be summable $D_{n \log n}$ and not Borel. Since the Borel definition does not include C_r, such series exist.

5. SOME INDICATIONS OF THE RELATION BETWEEN THE LEROY DEFINITION AND C_r.

We have attempted, by the method used in section 2 for the proof that $D_{n \log n}$ includes C_r, to show that the LeRoy definition includes C_r. To do this, it would be sufficient to prove

$$1.\ \lim_{t \to 1} \frac{\Gamma(nt+1)}{\Gamma(n+1)} = 1,$$

$$2.\ \lim_{n \to \infty} n^r \frac{\Gamma(nt+1)}{\Gamma(n+1)} = 0, \quad t < 1,$$

$$3.\ \sum_{n=0}^{\infty} n^r \left| \Delta^{r+1} \frac{\Gamma(nt+1)}{\Gamma(n+1)} \right|$$

converges for each $t < 1$ and is bounded for all $t < 1$.

It is easily seen that the first condition is satisfied. By using Stirling's formula, condition 2 can be shown to be satisfied for all values of r.

In order to show that condition 3 is satisfied, it would be sufficient to show that conditions 1′ and 2 of Lemma 1, section 2, are satisfied.

For the case $r = 1$, we shall prove that condition 1′ is satisfied. To show this, it is necessary to prove that

$$\sum_{k=0}^{n} k\Delta^2 \frac{\Gamma(kt+1)}{\Gamma(k+1)}$$

is bounded for all n and all t, $0 < t < 1$.

If we write $f_k(t)$ for $\dfrac{\Gamma(kt+1)}{\Gamma(k+1)}$, we have

$$S_n(t) = \sum_{k=0}^{n-1} k\Delta^2 f_k(t) = f_1(t) - nf_n(t) + (n-1)f_{n+1}(t);$$

analogous to formula (5), section 2. The first term of $S_n(t)$ is independent

* Hardy and Riesz, loc. cit., Theorem 9, page 9.

of n and bounded for all t under consideration. Therefore, we need consider only the last two terms.

Replacing $f_n(t)$ by $\dfrac{\Gamma(nt+1)}{\Gamma(n+1)}$, the last two terms of $S_n(t)$ give

$$S_n'(t) = (n-1)\frac{\Gamma(\overline{n+1}\,t+1)}{\Gamma(n+2)} - \frac{n\Gamma(nt+1)}{\Gamma(n+1)}$$

$$= \frac{(n-1)t\Gamma(\overline{n+1}t)}{n\Gamma(n)} - \frac{nt\Gamma(nt)}{\Gamma(n)}.$$

Since both terms of $S_n'(t)$ are bounded for $n \leqq n_0$, $t \leqq t_0 < 1$, it is necessary only to consider what happens to $S_n'(t)$ when n is large and t near 1.

We can write

$$S_n'(t) = \frac{\left(1-\dfrac{1}{n}\right)t\Gamma(\overline{n+1}t) - nt\Gamma(nt)}{\Gamma(n)}.$$

Applying Stirling's formula to the gamma functions, we have

$$S_n'(t) = n^{-n\delta+1}t^{nt+(1/2)}e^{n\delta+O(1/n)}$$
$$\times\left[\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{nt+t-(1/2)} n^{-\delta}t^te^{-t+O(1/n)} - 1\right],$$

where $\delta = 1 - t$.

If we write each term as an exponential and expand the logarithms into series, we get

(1) $$t^{nt+(1/2)}e^{n\delta+O(1/n)} = e^{-(1/2)\delta+O(n\delta^2)+O(1/n)} = e^{O(n\delta^2)}\left[1+O(\delta)+O\left(\frac{1}{n}\right)\right],$$

(2) $$\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{nt+t-(1/2)} e^{-t+O(1/n)} = \left[1+O\left(\frac{1}{n}\right)\right],$$

and

(3) $$t^t = 1 + O(\delta).$$

By the law of the mean and the fact that $\delta > 0$, we have

$$0 < 1 - n^{-\delta} < \delta \log n,$$

so that

(4) $$n^{-\delta} = 1 + O(\delta \log n).$$

Substituting (1), (2), (3), (4), in $S_n'(t)$, we get

$$S_n'(t) = n^{-n\delta+1}e^{O(n\delta^2)}\left[O\left(\frac{1}{n}\right) + O(\delta \log n)\right]$$
$$= e^{-n\delta \log n + O(n\delta^2)}[O(1) + O(n\delta \log n)].$$

But

$$O(n\delta^2) = [n\delta \log n] O\left(\frac{\delta}{\log n}\right) < \tfrac{1}{2} n\delta \log n$$

for n sufficiently large and t near 1. Therefore,

$$e^{O(n\delta^2)} < e^{(1/2) n\delta \log n}$$

and

$$\begin{aligned} S_n'(t) &= e^{-(1/2) n\delta \log n}[O(1) + O(n\delta \log n)] \\ &= O(e^{-(1/2) n\delta \log n}) + O(n\delta e^{-(1/2) n\delta \log n} \log n). \end{aligned}$$

Let $y = \frac{1}{2} n\delta \log n$, then

$$S_n'(t) = O(e^{-y}) + O(ye^{-y}).$$

Since y is positive, $S_n'(t)$ is bounded for all n and all $t < 1$; and condition 1′ of the lemma is satisfied.

It seems not unlikely that this proof can be carried through for higher values of r. However, in order to complete the proof, even for the case $r = 1$, it would be necessary to show that condition 2 of the lemma is satisfied. We have not been able to do this. We are unable to determine accurately the number of changes of sign for $\Delta^2 \dfrac{\Gamma(nt+1)}{\Gamma(n+1)}$. We succeeded, however, by drawing graphs for $\dfrac{\Gamma(tx)}{\Gamma(x)}$ for various values of t near 1, in making it appear very likely that $\dfrac{d^2}{dx^2}\left[\dfrac{\Gamma(tx)}{\Gamma(x)}\right]$ changes sign just once for t near 1.* While this does not constitute a proof, there seems but little doubt that LeRoy's definition includes C_1, and probably C_r. This belief is somewhat strengthened by the fact that Hardy's definition,

$$\lim_{t \to 0} \sum_{n=0}^{\infty} e^{-nt \log n} u_n,$$

which he considered "substantially equivalent" † to LeRoy's, includes C_r.‡

* For $t = .9$, the second derivative appeared to change sign for a value of x near 3.

† See Introduction.

‡ Our proof that $D_{n \log n}$ includes C_r can be carried through with but minor changes for Hardy's definition.

MODULAR INVARIANTS OF A BINARY GROUP WITH COMPOSITE MODULUS.

By Constance R. Ballantine.

1. Let $\mathfrak{G}$ be the group of transformations

$$\mathfrak{T}: \begin{cases} x' = ax + by, \\ y' = cx + dy, \end{cases}$$

also denoted by

$$\mathfrak{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

on the indeterminates x and y, where a, b, c, d are integers, and

$$\Delta = ad - bc$$

is prime to a given integer m. A rational integral function, $F(x, y)$, with integral coefficients, will be called an invariant of $\mathfrak{G}$ modulo m if

$$F(x', y') \equiv \Delta^{\lambda} F(x, y) \pmod{m},$$

identically in x, y, a, b, c, d, after x', y', Δ have been replaced by their expressions above.

We shall frequently find it convenient to work, not with the infinite group $\mathfrak{G}$, but with the finite group, denoted by $\mathfrak{G}'$, of *classes* $\mathfrak{T}'$ of all transformations congruent to a given $\mathfrak{T}$ modulo m. We shall write

$$\mathfrak{T}_1 \equiv \mathfrak{T} \pmod{m},$$

if

$$a_1 \equiv a, \quad b_1 \equiv b, \quad c_1 \equiv c, \quad d_1 \equiv d \pmod{m}.$$

Thus, if $\mathfrak{T}_1$ belongs to $\mathfrak{T}'$,

$$F(a_1x + b_1y,\ c_1x + d_1y) \equiv F(ax + by,\ cx + dy) \pmod{m},$$

identically in x and y. Hence we may state

Theorem I. *The invariance of a rational integral function with integral coefficients under the finite group $\mathfrak{G}'$ modulo m is equivalent to its invariance under $\mathfrak{G}$ modulo m.*

2. Professor L. E. Dickson* has treated the case $m = p$, a prime. Restricting attention at first to the group H (with corresponding finite group H') for which

$$\Delta \equiv 1 \pmod{p},$$

he has shown later that the fundamental system of invariants is the same

* *Madison Colloquium Lectures*, pp. 33 ff.

as when Δ is merely prime to p. This fundamental system he has found to be

$$L = x^p y - x y^p,$$
$$Q = \frac{x^{p^2} y - x y^{p^2}}{L}.$$

The group H consists of all transformations

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \equiv 1 \pmod{p};$$

where a, b, c, d are integers; and the finite group H' is the group of classes S' of all transformations

$$S \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod{p},$$

where $\alpha, \beta, \gamma, \delta$ are integers reduced modulo p.

3. Passing now to the case of a composite modulus

$$\pi = p_1 p_2 \cdots p_n,$$

where the p_i are distinct primes, we wish to find all rational integral functions $I(x, y)$ with integral coefficients which are invariant modulo π under the group Γ of transformations

$$\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \equiv 1 \pmod{\pi},$$

where a, b, c, d are integers; i.e., all functions $I(x, y)$ for which

$$I(ax + by,\ cx + dy) \equiv I(x, y) \pmod{\pi},$$

identically in x and y, for all admissible a, b, c, d. By Theorem I, this is equivalent to finding all the invariants, modulo π, of the group Γ' of classes Θ' of transformations

$$\Theta \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod{\pi},$$

where $\alpha, \beta, \gamma, \delta$ are integers reduced modulo π.

4. Let p denote any one of the p_i, and m the product of the others, so that $\pi = pm$. Consider the sub-group G' of Γ', composed of all classes of transformations of Γ which are congruent to identity modulo m. These transformations, taken altogether, form the corresponding infinite sub-group G, but it is with the finite sub-group that we shall chiefly be dealing. Every class of transformations of G' is composed of transformations

$$T \equiv \begin{pmatrix} \alpha & m\beta \\ m\gamma & \delta \end{pmatrix} \pmod{\pi}, \qquad \alpha \equiv \delta \equiv 1 \pmod{m},$$

where α, $m\beta$, $m\gamma$, δ are integers reduced modulo π; while, since T belongs to Γ, we have

$$\alpha\delta - m^2\beta\gamma \equiv 1 \pmod{\pi}.$$

The latter congruence, being satisfied identically modulo m, requires merely that

$$\alpha\delta - m^2\beta\gamma \equiv 1 \pmod{p}.$$

Since α and δ are residues of integers modulo π and are congruent to unity modulo m, they range over the values

$$1,\ m+1,\ 2m+1,\ \cdots,\ (p-1)m+1,$$

which are congruent in some order to

$$0,\ 1,\ 2,\ \cdots,\ p-1,$$

modulo p. The same is true of the integers $m\beta$ and $m\gamma$, which range over the values

$$0,\ m,\ 2m,\ \cdots,\ (p-1)m.$$

Thus to every T' (class of transformations congruent to a given T modulo π) there corresponds an S' of H', *viz.*, the class of transformations

$$S \equiv \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \qquad \alpha'\delta' - \beta'\gamma' \equiv 1 \pmod{p},$$

where

(1) $$\alpha' \equiv \alpha, \quad \beta' \equiv m\beta, \quad \gamma' \equiv m\gamma, \quad \delta' \equiv \delta \pmod{p},$$

are integers reduced modulo p, and, conversely, to every such S' corresponds a T'. For we can solve the congruences

(2) $$\begin{cases} \alpha \equiv \alpha', \quad m\beta \equiv \beta', \quad m\gamma \equiv \gamma', \quad \delta \equiv \delta' \pmod{p}, \\ \alpha \equiv \delta \equiv 1 \pmod{m}, \end{cases}$$

for integers α, $m\beta$, $m\gamma$, δ, reduced modulo π, and, since

$$\alpha\delta - m^2\beta\gamma \equiv 1 \pmod{p},$$

by the definition of S' and (2_1), while the same congruence holds modulo m by (2_2), it holds modulo π, and all transformations

$$T \equiv \begin{pmatrix} \alpha & m\beta \\ m\gamma & \delta \end{pmatrix} \pmod{\pi}$$

are in Γ. Further, this correspondence is one-to-one, since the solution of (1) in integers reduced modulo p is unique, and the same is true of the solution of (2) in integers reduced modulo π.

Also, given any two classes, T_1', T_2', of G', composed of all transformations

$$T_1 \equiv \begin{pmatrix} \alpha_1 & m\beta_1 \\ m\gamma_1 & \delta_1 \end{pmatrix}, \qquad T_2 \equiv \begin{pmatrix} \alpha_2 & m\beta_2 \\ m\gamma_2 & \delta_2 \end{pmatrix} \pmod{\pi},$$

respectively, where, from the definition of G,

$$\alpha_1 \equiv \delta_1 \equiv \alpha_2 \equiv \delta_2 \equiv 1 \pmod{m},$$

we have

$$\begin{aligned} T_3 &\equiv T_1 T_2 \\ &\equiv \begin{pmatrix} \alpha_1\alpha_2 + m^2\beta_1\gamma_2 & m\alpha_1\beta_2 + m\beta_1\delta_2 \\ m\gamma_1\alpha_2 + m\delta_1\gamma_2 & m^2\gamma_1\beta_2 + \delta_1\delta_2 \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha_3 & m\beta_3 \\ m\gamma_3 & \delta_3 \end{pmatrix} \pmod{\pi}. \end{aligned}$$

The product of two transformations S_1, S_2, belonging to the classes S_1', S_2', corresponding in the manner described to T_1', T_2', respectively, is

$$\begin{aligned} S_3 &\equiv S_1 S_2 \\ &\equiv \begin{pmatrix} \alpha_1' & \beta_1' \\ \gamma_1' & \delta_1' \end{pmatrix} \begin{pmatrix} \alpha_2' & \beta_2' \\ \gamma_2' & \delta_2' \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha_1'\alpha_2' + \beta_1'\gamma_2' & \alpha_1'\beta_2' + \beta_1'\delta_2' \\ \gamma_1'\alpha_2' + \delta_1'\gamma_2' & \gamma_1'\beta_2' + \delta_1'\delta_2' \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha_3' & \beta_3' \\ \gamma_3' & \delta_3' \end{pmatrix} \pmod{p}, \end{aligned}$$

where α_3', β_3', γ_3', δ_3' are the residues of α_3, $m\beta_3$, $m\gamma_3$, δ_3 modulo p, and hence S_3' is the class of H' corresponding to T_3' of G'.

Since p was any one of the p_i, we have

THEOREM II. *Each of the n sub-groups G_i' of the group Γ' of classes of transformations with determinant congruent to unity modulo π, viz., the group of all classes of Γ' composed of transformations which are congruent to identity modulo $m_i = \pi/p_i$, is isomorphic with the group H_i' of classes of transformations with determinant congruent to unity modulo p_i.*

5. We shall now prove the following

LEMMA. *For $i < n$, let T_1', T_2', $\cdots$, T_i' range over all the classes of G_1', G_2', $\cdots$, G_i', respectively. Then the products $T_1'T_2' \cdots T_i'$ are all distinct and form the sub-group J_i' of classes U_i' of transformations U_i of Γ which are congruent to identity modulo $l_i = \pi/(p_1p_2 \cdots p_i)$.*

This is true by definition when $i = 1$; i.e., $l_1 = m_1$. Suppose it true when the above i is replaced by $i - 1$. Then, first, the groups J_{i-1}' and G_i' have no class in common save that of the transformations congruent to identity (I) modulo π. For suppose any

$$U_{i-1}' = T_i'.$$

Then every

$$U_{i-1} \equiv T_i \pmod{\pi}.$$

But

$$T_i \equiv I \pmod{m_i},$$

and

$$U_{i-1} \equiv I \pmod{l_{i-1}},$$

and hence modulo p_i, a divisor of both l_{i-1} and π. Thus

$$T_i \equiv U_{i-1} \equiv I \pmod{p_i},$$

and, since m_i is prime to p_i and their product is π, we get

$$T_i \equiv I \pmod{\pi}.$$

Further, the products $U'_{i-1}T'_i$ are all distinct, where U'_{i-1}, T'_i range over all the classes of J'_{i-1}, G'_i, respectively. For, if such a product be equal to a second product (in which the letters are distinguished by an asterisk), i.e., if

$$U'_{i-1}T'_i = \overset{*}{U}'_{i-1}\overset{*}{T}'_i,$$

then every

$$U_{i-1}T_i \equiv \overset{*}{U}_{i-1}\overset{*}{T}_i \pmod{\pi},$$

and

$$\overset{*}{U}{}^{-1}_{i-1}U_{i-1} \equiv \overset{*}{T}_i T_i^{-1} \pmod{\pi},$$

whence each member of the last congruence is congruent to identity modulo π, and

$$\overset{*}{U}_{i-1} \equiv U_{i-1}, \qquad \overset{*}{T}_i \equiv T_i \pmod{\pi},$$

which implies

$$\overset{*}{U}'_{i-1} = U'_{i-1}, \quad \overset{*}{T}'_i = T'_i.$$

We note that $l_{i-1} = p_i l_i$, and that m_i is also divisible by l_i. Thus, since every

$$U_{i-1} \equiv I \pmod{l_{i-1}},$$

it satisfies the same congruence modulo l_i; and, similarly, every

$$T_i \equiv I \pmod{m_i},$$

and hence also modulo l_i. Therefore we have every product

$$U_{i-1}T_i \equiv I \pmod{l_i},$$

whence the products belong to J_i, and the product classes to J'_i.

Conversely, given the class U'_i of transformations

$$U_i \equiv \begin{pmatrix} \alpha & l_i\beta \\ l_i\gamma & \delta \end{pmatrix} \pmod{\pi},$$

where

$$\alpha \equiv \delta \equiv 1 \pmod{l_i},$$

we can find integers α_{i-1}, $l_{i-1}\beta_{i-1}$, $l_{i-1}\gamma_{i-1}$, δ_{i-1}, α_i, $m_i\beta_i$, $m_i\gamma_i$, δ_i, reduced modulo π such that

$$\alpha_{i-1}\alpha_i \equiv \alpha, \qquad \delta_{i-1}\delta_i \equiv \delta,$$

$$m_i\alpha_{i-1}\beta_i + l_{i-1}\beta_{i-1}\delta_i \equiv \beta, \qquad l_{i-1}\gamma_{i-1}\alpha_i + m_i\delta_{i-1}\gamma_i \equiv \gamma \pmod{\pi},$$

while

$$\alpha_{i-1} \equiv \delta_{i-1} \equiv 1 \pmod{l_{i-1}},$$

$$\alpha_i \equiv \delta_i \equiv 1 \pmod{m_i},$$

so that, since

$$l_{i-1}m_i \equiv 0 \pmod{\pi},$$

we can factor each U_i, modulo π, into the product of

$$U_{i-1} \equiv \begin{pmatrix} \alpha_{i-1} & l_{i-1}\beta_{i-1} \\ l_{i-1}\gamma_{i-1} & \delta_{i-1} \end{pmatrix} \pmod{\pi},$$

and

$$T_i \equiv \begin{pmatrix} \alpha_i & m_i\beta_i \\ m_i\gamma_i & \delta_i \end{pmatrix} \pmod{\pi}.$$

Thus all the classes U'_i of J'_i are expressible as products of classes U'_{i-1} of J'_{i-1}, T'_i of G'_i, and J'_i is obtainable by composition from J'_{i-1} and G'_i.

If i is taken to be n, the first part of the lemma is still true, thus all the products $T'_1T'_2 \cdots T'_n$ are distinct, where $T'_1, T'_2, \cdots, T'_n$ range over all the classes of $G'_1, G'_2, \cdots, G'_n$, respectively. The order of Γ' is the product of the orders of the sub-groups $G'_1, G'_2, \cdots, G'_n$. For, given any class Θ' of transformations

$$\Theta \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod{\pi},$$

where $\alpha, \beta, \gamma, \delta$ are integers reduced modulo π, we can determine uniquely for each value of i a class S'_i of transformations

$$S_i \equiv \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \qquad \alpha_i\delta_i - \beta_i\gamma_i \equiv 1 \pmod{p_i},$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are integers reduced modulo p_i, such that

$$\alpha_i \equiv \alpha, \quad \beta_i \equiv \beta, \quad \gamma_i \equiv \gamma, \quad \delta_i \equiv \delta \pmod{p_i};$$

and, conversely, given any set $S'_1, S'_2, \cdots, S'_n$ we can solve

$$\alpha \equiv \alpha_i, \quad \beta \equiv \beta_i, \quad \gamma \equiv \gamma_i, \quad \delta \equiv \delta_i \pmod{p_i}$$

$(i = 1, 2, \cdots, n)$ for $\alpha, \beta, \gamma, \delta$, for which $\alpha\delta - \beta\gamma \equiv 1 \pmod{\pi}$ uniquely among the residues of integers modulo π, so that every set $S'_1, S'_2, \cdots, S'_n$ determines a class Θ' of Γ'. Thus the order of Γ' is the product of the orders of the H'_i, which equal those of the G'_i, and we have

THEOREM III. *The total group Γ' of classes of transformations with determinant congruent to unity modulo π is obtainable by composition of the n sub-groups G'_i, each composed of those classes of Γ' whose transformations are congruent to identity modulo $m_i = \pi/p_i$.*

6. Let $I(x, y)$ be any rational integral function with integral coefficients which is invariant under Γ modulo π. By Theorem I, this is equivalent to the invariance of $I(x, y)$ under the group Γ', i.e.,

$$I(\alpha x + \beta y, \gamma x + \delta y) \equiv I(x, y) \pmod{\pi},$$

for every set $\alpha, \beta, \gamma, \delta$, of integers reduced modulo π, for which

$$\alpha\delta - \beta\gamma \equiv 1 \pmod{\pi}.$$

In particular, $I(x, y)$ is invariant under every class T'_i of transformations

$$T_i \equiv \begin{pmatrix} \alpha_i & m_i\beta_i \\ m_i\gamma_i & \delta_i \end{pmatrix} \pmod{\pi},$$
$$\alpha_i \equiv \delta_i \equiv 1 \pmod{m_i},$$

of G'_i $(i = 1, 2, \cdots, n)$, and thus under the corresponding class S'_i of transformations

$$S_i \equiv \begin{pmatrix} \alpha'_i & \beta'_i \\ \gamma'_i & \delta'_i \end{pmatrix}, \qquad \alpha'_i\delta'_i - \beta'_i\gamma'_i \equiv 1 \pmod{p_i},$$

where

$$\alpha'_i \equiv \alpha_i, \quad \beta'_i \equiv m_i\beta_i, \quad \gamma'_i \equiv m_i\gamma_i, \quad \delta'_i \equiv \delta_i \pmod{p_i},$$

are integers reduced modulo p_i, of H'_i $(i = 1, 2, \cdots, n)$. By the isomorphism proved in Theorem II, when T'_i ranges over all the classes of G'_i, S'_i ranges over all the classes of H'_i, and thus $I(x, y)$ is invariant under all the transformations of H_i modulo p_i $(i = 1, 2, \cdots, n)$.

Conversely, suppose $I(x, y)$ is a rational integral invariant of any one of the H_i modulo p_i, i.e., invariant under every class S'_i of transformations

$$S_i \equiv \begin{pmatrix} \alpha'_i & \beta'_i \\ \gamma'_i & \delta'_i \end{pmatrix}, \qquad \alpha'_i\delta'_i - \beta'_i\gamma'_i \equiv 1 \pmod{p_i}.$$

Then, since we can solve

$$\alpha_i \equiv \alpha'_i, \quad m_i\beta_i \equiv \beta'_i, \quad m_i\gamma_i \equiv \gamma'_i, \quad \delta_i \equiv \delta'_i \pmod{p_i},$$
$$\alpha_i \equiv \delta_i \equiv 1 \pmod{m_i},$$

for integers $\alpha_i, m_i\beta_i, m_i\gamma_i, \delta_i$, reduced modulo π, $I(x, y)$ is invariant under the corresponding class T'_i of transformations

$$T_i \equiv \begin{pmatrix} \alpha_i & m_i\beta_i \\ m_i\gamma_i & \delta_i \end{pmatrix} \pmod{\pi}$$

of G'_i modulo π, and thus, using again the isomorphism between G'_i and H'_i, under the sub-group G'_i modulo π. Since, by Theorem III, the sub-groups G'_i $(i = 1, 2, \cdots, n)$ generate the total group Γ' modulo π, if any rational integral function with integral coefficients is invariant under every H'_i modulo p_i $(i = 1, 2, \cdots, n)$, it is an invariant of Γ', and hence of Γ, modulo π.

THEOREM IV. *A necessary and sufficient condition for the invariance of a rational integral function $I(x, y)$ with integral coefficients under the group Γ of transformations with determinant congruent to unity modulo $\pi = p_1 p_2 \cdots p_n$, is that $I(x, y)$ be invariant under every group H_i of transformations with determinant congruent to unity modulo p_i $(i = 1, 2, \cdots, n)$.*

7. As mentioned in § 2, a fundamental system of invariants of H_i modulo p_i is given by

$$L_i = x^{p_i} y - x y^{p_i},$$
$$Q_i = \frac{x^{p_i^2} y - x y^{p_i^2}}{L_i}.$$

Thus every rational integral invariant of H_i is congruent, modulo p_i, to a polynomial in L_i and Q_i with integral coefficients, or we may say that every invariant is of the form

$$F(L_i, Q_i) + p_i f(x, y),$$

where F and f are polynomials with integral coefficients.

By Theorem IV, if $I(x, y)$ is an invariant of Γ modulo π, it is an invariant of every H_i modulo p_i, thus it satisfies the equations

$$I(x, y) = F_i(L_i, Q_i) + p_i f_i(x, y), \tag{1}$$

$(i = 1, 2, \cdots, n)$. Since the greatest common divisor of the numbers m_i $(i = 1, 2, \cdots, n)$ is unity, there exist integers k_i such that

$$\sum_{i=1}^{n} k_i m_i = 1.$$

Multiplying each of the equations (1) by the corresponding $k_i m_i$, and adding, we have

$$I(x, y) = \sum_{i=1}^{n} k_i m_i F_i(L_i, Q_i) + \pi \sum_{i=1}^{n} k_i f_i(x, y),$$

whence

$$I(x, y) \equiv \sum_{i=1}^{n} k_i m_i F_i(L_i, Q_i) \pmod{\pi}.$$

As $k_i F_i(L_i, Q_i)$ is an invariant modulo p_i of H_i, we have, finally,

THEOREM V. *Every invariant of the group Γ of transformations with determinant congruent to unity modulo $\pi = p_1 p_2 \cdots p_n$ is a sum of expressions, each of which is the product of an $m_i = \pi / p_i$ by an invariant of the corresponding group H_i of transformations with determinant congruent to unity modulo p_i; and, conversely, every such product is an invariant of Γ modulo π by definition.*

DETERMINATION OF A SURFACE BY ITS CURVATURES AND SPHERICAL REPRESENTATION.

By W. C. Graustein.

1. **Introduction.**—The principal problem of this paper is to ascertain in how far a surface is determined by its curvatures and the directions of its normals,* or, stated more broadly, to find conditions, necessary and sufficient, that a definite, positive, differential quadratic form in the variables u, v and two functions, $K(u, v)$ and $\overline{K}(u, v)$, be respectively the third fundamental form and the total and mean curvatures of a surface, and to ascertain, when these conditions are fulfilled, the number of surfaces determined.†

The conditions that the given elements, in their prescribed capacities, determine at least one surface, though not easy to obtain in general form, are readily found when appropriate parameters are introduced.

Developable surfaces are evidently excluded by hypothesis. It is found that a non-developable surface is in general uniquely‡ determined by the prescribed elements. In the exceptional cases there is either a pair or a one-parameter family of surfaces. The surfaces of a pair or two surfaces of a family we shall call *cross-congruent.*

It is geometrically evident that the surfaces parallel respectively to those of a pair, or family, of cross-congruent surfaces and at the same algebraic distance, c, from them, also constitute a pair, or family, of cross-congruent surfaces. We shall call the two pairs, or families, *parallel.* If the ratio of the curvatures of the surfaces of the given pair, or family,

* This problem is to a certain extent analogous to the problem, proposed and solved by Bonnet, of determining the surfaces applicable to a given surface with preservation of both curvatures. Cf. Bonnet, "Mémoire sur la théorie des surfaces applicables sur une surface donneé," *Journal de l'École Polytechnique*, Vol. 42 (1867), pp. 72, foll.; also, Author, "Applicability with preservation of both curvatures," *Bull. Amer. Math. Soc.*, Vol. 30 (1924), where new results bearing on Bonnet's problem are derived by the methods developed in the present paper.

† Equivalent to the prescription of K and $\overline{K}$ is that of the principal normal curvatures, $1/r_1$ and $1/r_2$, provided that their order is left arbitrary. The sign of $\overline{K}$, or those of r_1 and r_2, are immaterial, since they depend on the orientation of the surface normal; however, if two surfaces satisfy the same prescribed conditions except perhaps that their mean curvatures are opposite in sign, we agree to orient their normals so that the mean curvatures will be equal.

‡ We adopt the usual convention of thinking of a surface as unique when it is determined except for rigid motions and reflections.

is $\bar{K}/K = r_1 + r_2$, that of the surfaces of the parallel pair, or family, is $\bar{K}/K - 2c$.

A family of associate minimal surfaces is a family of cross-congruent surfaces. So then is any parallel family. Families of these two types we shall call *families of associate surfaces for which $\bar{K}/K = const.$* Both they and the function $\bar{K}/K$ play special rôles.

The lines of curvature of the surfaces of a pair are never represented on the sphere by isometric systems, whereas those of the surfaces of a family always are. In the latter case, the ∞^1 isometric systems on the sphere make constant angles with one another only in the case of associate surfaces for which $\bar{K}/K =$ const.

The problem of determining whether or not two given surfaces are cross-congruent can be handled by the method used by Bonnet in treating the corresponding problem of Minding. The details are left to the reader. We note merely that, in imposing the condition that the curvatures be preserved, it is convenient to use, not K and $\bar{K}$, but K and $\bar{K}/K$. Our primary interest in this investigation consists in ascertaining under what conditions two surfaces are cross-congruent in a continuous infinity of ways. In stating these conditions we can assume that the surfaces are cross-congruent and are referred to the same parameters.

THEOREM 1. *Two cross-congruent surfaces are cross-congruent in a continuous infinity of ways if and only if their common curvatures, K and $\bar{K}$, are functionally dependent and the equation $\bar{K}/K = const.$ represents on the sphere an isometric family of geodesic parallels, or, if $\bar{K}/K$ is constant, the equation $K = const.$ represents such a family on the sphere.*

If $\bar{K}/K$ is not constant, the first condition of the theorem, namely, that K and $\bar{K}$ be functionally dependent, is, in the case of families of cross-congruent surfaces, by itself sufficient! Moreover, the differential system defining these families can be completely integrated and the point coordinates of the surfaces found by quadratures. When $\bar{K}/K$ is constant, no difficulties arise, inasmuch as cross-congruence, in the special case of minimal surfaces, is identical with applicability.

The second half of the paper deals with the problem from the point of view of parallel maps.* It is evident that either of two cross-congruent surfaces can be rotated and then, if necessary, reflected in a point, so that the normals in corresponding points will be parallel and similarly oriented. Hence *the problem of finding the surfaces cross-congruent to a given surface, if any exist, is identical with the problem of determining the surfaces in point correspondence with the given surface by parallel normals so that the total and*

* Cf. Author, "Parallel Maps of Surfaces," *Trans. Amer. Math. Soc.*, Vol. 23 (1922), pp. 298–332. This paper will be referred to as "Reference A."

mean curvatures in corresponding points are respectively equal. A parallel map of this type we shall call *cross-congruent.* Of course, it is assumed that the map is not equivalent to a rigid motion or a reflection.

A parallel map is non-parabolic or parabolic, according as there exist on the surfaces basic, i.e., corresponding, conjugate systems, or merely basic families of asymptotic lines.

As a fundamental result we note that there is at most one surface cross-congruent to a given surface by a parallel map for which the basic conjugate system, or family of asymptotic lines, is prescribed, save when the given surface is minimal and the prescribed conjugate system consists of its generators. In other words, the basic systems of the parallel maps of one of ∞^1 cross-congruent surfaces on the others are all distinct save in the case of a family of associate minimal surfaces.

A non-parabolic map is cross-congruent if and only if the normal curvatures of the two surfaces in the basic conjugate directions are equal crosswise, but not also directly, that is, if and only if each of these curvatures for the one surface is equal to the non-corresponding one, but not also to the corresponding one, on the other surface. Herein lies the motivation of the term "cross-congruent"; if the curvatures were directly equal, the surfaces would be congruent. Moreover, in a map of the type in question, the point invariants of the basic conjugate systems are equal crosswise.

A parabolic map is cross-congruent only when the two surfaces are suitably related ruled surfaces whose secondary asymptotic lines cut equal segments from the rulings.

Various particular questions of interest are taken up, for example, surfaces of revolution, surfaces of constant total curvature or of constant mean curvature, not zero, in the first part of the paper, and in the second part, special cases of cross-congruent non-parabolic maps, including those of translation surfaces, of associate surfaces of Bianchi, and those in which the basic conjugate systems have equal point invariants.

I. Direct Treatment.

2. **Existence theorems.**—Let the given definite, positive, differential quadratic form be

$$\mathcal{E}du^2 + 2\mathcal{F}dudv + \mathcal{G}dv^2, \tag{1}$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are real, analytic functions of the real variables u, v, and let the given functions, $K(u, v)$, $\overline{K}(u, v)$, be also real and analytic.

An evident necessary condition that there exist a surface, $x = x(u, v)$:

$$x_1 = x_1(u, v), \qquad x_2 = x_2(u, v), \qquad x_3 = x_3(u, v), \tag{2}$$

whose total and mean curvatures are K and $\overline{K}$ and whose third fundamental